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Solution to 2017-1 Problem 1

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Abstract. We give a construction of the figure in 2017-1 Problem 1 and give a general relationship of the radii of the circles in the diagram.

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Problem 1. *See* Figure 1*, where the three small circles seems to be congruent.*

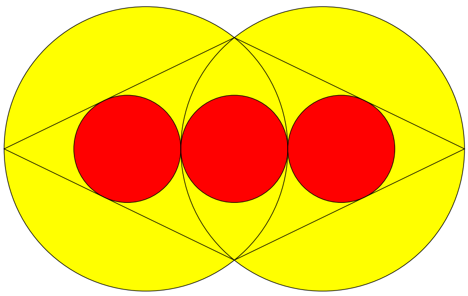

FIGURE 1. Proposed problem with no text.

We suppose that the large circles have radius R and are centered at A, A' such that $AA' = 2a$. The radius of the three small circles is then

$$r = R - a \tag{1}$$

(See Figure 2). First we find r in terms of R.

We assume that the line AA' meets one of the large circles in points C and E, where C does not lie on any small circle, O is the midpoint of AA', B is one of the points of intersection of the two large circles, and T is the point of tangency

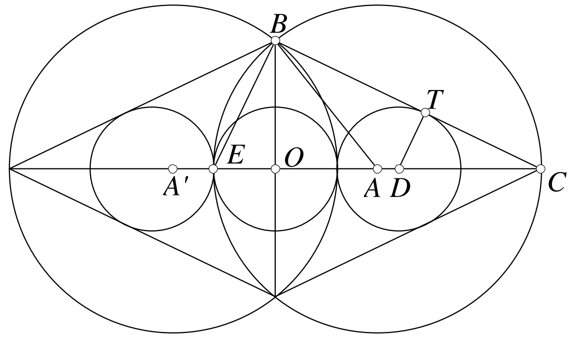

FIGURE 2. Solution.

of BC and one of the small circles. Since $OB^2 = OC \cdot OE = r(2R - r)$, and the triangles TDC and OBC are similar,

$$\frac{TD}{DC} = \frac{OB}{BC} \Rightarrow \frac{r}{2R - 3r} = \frac{\sqrt{r(2R-r)}}{\sqrt{r(2R-r) + (2R-r)^2}} = \sqrt{\frac{r}{2R}}, \tag{2}$$

giving the relation

$$9r^2 - 14rR + 4R^2 = 0,$$

from which we get r in terms of R (we look for $r < R$):

$$r = \frac{7 - \sqrt{13}}{9} R. \tag{3}$$

Eliminating r from (1) and (3), we get $R = (\sqrt{13} - 2)a$.

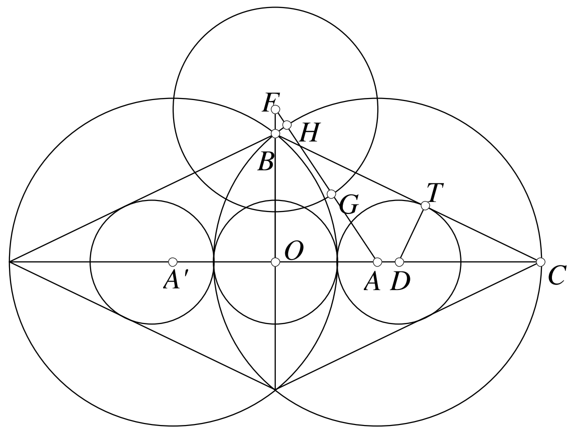

FIGURE 3. Construction.

Construction. Now we assume that a segment $AA' = 2a$ with midpoint O is given. We erect a perpendicular OF to OA equal to $\frac{3}{2}OA$ (see Figure 3). Now we find a point G on segment FA such that $FG = OA$ and the reflection H of A in G. Then $AH = (\sqrt{13} - 2)a$ holds. Let $R = AH$. We construct the circles (A, R) and (A', R), and the remaining parts of the figure.

Generalization. We can generalize Problem 1 by considering m circles between the two large circles and n small circles on each side.

Figure 4 shows the cases $m = 2, n = 3$ (left) and $m = 3, n = 2$ (right).

In this case (2) becomes

$$\frac{r}{2R-(2m+2n-1)r} = \frac{\sqrt{mr(2R-mr)}}{\sqrt{mr(2R-mr)+(2R-mr)^2}} = \sqrt{\frac{mr}{2R}}.$$

The last equation yields

$$4mR^2 - 2(4m^2+4mn-2m+1)rR + m(2m+2n-1)^2r^2 = 0$$

which gives

$$R = \frac{t_{m,n}+\sqrt{2t_{m,n}-1}}{4m}r,$$

where $t_{m,n} = 2m(2m+2n-1)+1$.

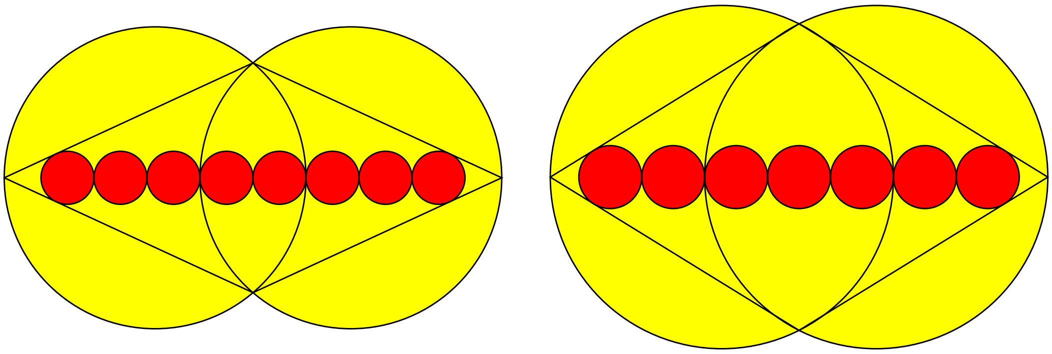

Figure 4. Generalization.