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A Note on a Pappus Sangaku Problem and a Family of Integer Sequences

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Abstract. We give a condition that a Pappus chain is related to certain integer sequences.

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There are many sangaku problems involving infinite chains of mutually tangent circles that, according to the western tradition, are named Pappus chains [4]. We could name these problems as *Pappus Sangaku* and here we consider one of them. Let us look at Figure 1 where two semi-circles C_a and C_b are tangent in O; together with the outer semi-circle C_{out}, tangent to both of them, they form a geometrical figure named *arbelos*.

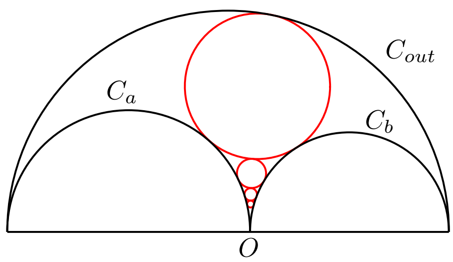

Figure 1.

In this paper we consider a Pappus chain consisting of the circles tangent to C_a and C_b and we put them in relation with certain integer sequences. By taking a cue from [2, 3], the radius of the n-th circle of the chain is expressed by

$$r_n = \frac{ab(a+b)}{[n(a+b)]^2 - ab},$$

where a and b are the radii of C_a and C_b, respectively, and the first circle of the chain is the incircle of the arbelos.

Let us consider the ratio τ_n between the outer circle radius $a+b$ and the radius of the n-th circle of the chain; by introducing the parameter $\lambda = \frac{a}{b} > 0$, one has:

$$\tau_n = \frac{(\lambda+1)^2}{\lambda} n^2 - 1.$$

Therefore $\{\tau_n\}$ is an integer sequence if and only if $(\lambda+1)^2/\lambda = J,\ J \in \mathbb{N}^+$. Solving the last equation for λ one obtains:

$$\lambda = \frac{-2 + J \pm \sqrt{J^2 - 4J}}{2}, \qquad J \geq 4. \tag{1}$$

Note that both the solutions are acceptable for τ_n being an integer; in fact, one is the reciprocal of the other and the solution corresponding to the plus sign holds when $a > b$; also the solution with minus sign holds when $a < b$. Anyway, both the values for λ lead to the same integer sequence.

Depending on the value of J, several integer sequences classified in OEIS (On Line Encyclopedia of Integer Sequences) can be found [1]. Notice that $\lambda = \Phi^{\pm 2}$ if $J = 5$ by (1), while $\lambda = \Phi^{\pm 4}$ if $J = 9$, where $\Phi = \frac{\sqrt{5}+1}{2}$, is the *golden number* or the *golden ratio*.

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