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A note on Pappus chain and a collinear theorem

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Abstract. We give a collinear theorem for Pappus chain.

Keywords. Pappus chain, collinearity

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For the Pappus sangaku diagram arising from an arbelos, relationships between the radii of the circles in the chain have been mostly considered, which is similar to Wasan geometry. Contrarily, in this note we consider a collinear theorem for the diagram.

We consider an arbelos with incircle δ formed by the three semicircles α, β and γ with diameters BC, CA and AB, respectively for a point C on the segment AB (see Figure 1). Let a and b be the radii of the semicircles α and β, respectively. We use a rectangular coordinate system with origin C such that A and B have coordinates $(-2b, 0)$ and $(2a, 0)$, respectively, and the three semicircles are constructed in the region $y \geq 0$.

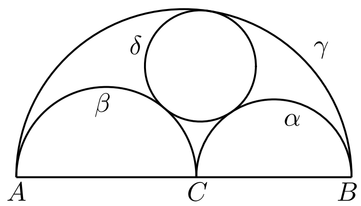

Figure 1.

Let $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} = \{\alpha, \beta, \gamma\}$ and $c = a + b$. We consider the chain of circles $\{\delta = \delta_1, \delta_2, \delta_3, \cdots\}$ whose members touch the circles ε_2 and ε_2. If $\varepsilon_1 = \alpha$, the chain is denoted by $\mathcal{C}_\alpha$. The chains $\mathcal{C}_\beta$ and $\mathcal{C}_\gamma$ are defined similarly (see Figure 2). Let (x_n, y_n) and r_n be the coordinates of the center and the radius of the circle δ_n. Then $y_n = 2nr_n$ holds by Pappus chain theorem, and x_n and r_n are given in Table 1 [1, 2].

Chain	x_n	r_n
$\mathcal{C}_\alpha$	$-2b+\dfrac{bc(b+c)}{n^2a^2+bc}$	$\dfrac{abc}{n^2a^2+bc}$
$\mathcal{C}_\beta$	$2a-\dfrac{ca(c+a)}{n^2b^2+ca}$	$\dfrac{abc}{n^2b^2+ca}$
$\mathcal{C}_\gamma$	$\dfrac{ab(b-a)}{n^2c^2-ab}$	$\dfrac{abc}{n^2c^2-ab}$

Table 1.

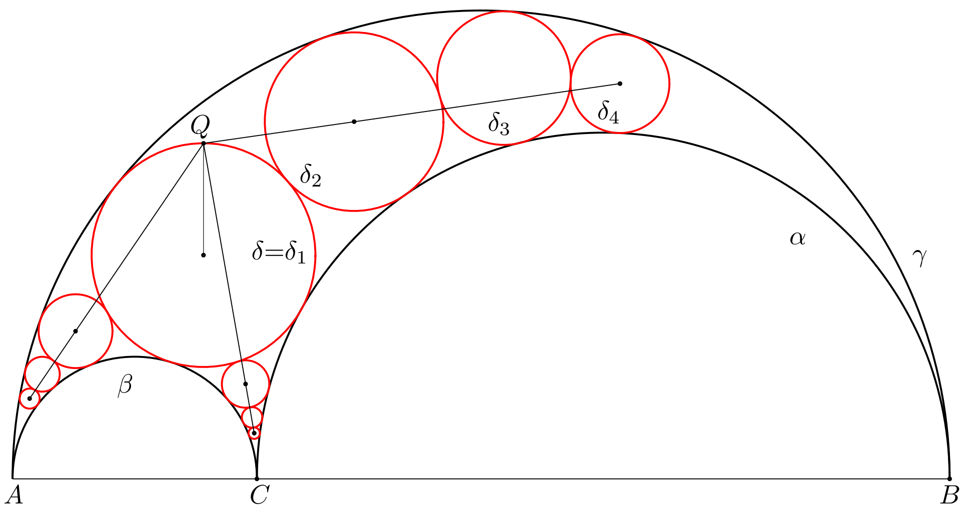

Figure 2: $\mathcal{C}_\beta$, $\varepsilon_1=\beta$, $\{\varepsilon_2,\varepsilon_3\}=\{\gamma,\alpha\}$

Theorem 1. *The farthest point on δ from AB, the centers of δ_2 and δ_4 are collinear.*

Proof. Let Q be the farthest point on δ from AB. Then Q has coordinates (x_1, y_1+r_1). We consider the chain $\mathcal{C}_\alpha$. If $k=ab^2c^2(b+c)/((a^2+bc)(4a^2+bc)(16a^2+bc))$,

$$\begin{vmatrix} x_1 & y_1+r_1 & 1 \\ x_2 & y_2 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} = \begin{vmatrix} -2b+\dfrac{bc(b+c)}{a^2+bc} & \dfrac{3abc}{a^2+bc} & 1 \\ -2b+\dfrac{bc(b+c)}{4a^2+bc} & \dfrac{4abc}{4a^2+bc} & 1 \\ -2b+\dfrac{bc(b+c)}{16a^2+bc} & \dfrac{8abc}{16a^2+bc} & 1 \end{vmatrix}$$

$$= k\begin{vmatrix} 1 & 3 & a^2+bc \\ 1 & 4 & 4a^2+bc \\ 1 & 8 & 16a^2+bc \end{vmatrix} = k\begin{vmatrix} 1 & 3 & a^2+bc \\ 0 & 1 & 3a^2 \\ 0 & 4 & 12a^2 \end{vmatrix} = k\begin{vmatrix} 1 & 3a^2 \\ 4 & 12a^2 \end{vmatrix} = 0.$$

Therefore Q and the centers of δ_2 and δ_4 are collinear. The rest of the theorem can be proved similarly. □

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