Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 2 (2018), pp.15-16
Received 14 May 2018. Published on-line 10 June 2018
web: http://www.sangaku-journal.eu/


# Solution to 2018-1 Problem 1

Viera Čerňanová
Trnava University, Faculty of Education,
Department of Mathematics and Computer Science,
Priemyselná 4, 918 43 Trnava, Slovakia
e-mail:vieracernanova@hotmail.com

**Abstract.** We give a solution to 2018-1 Problem 1.

**Keywords.** sangaku, square, equilateral triangle, incircle.

**Mathematics Subject Classification (2010).** 51M04.

**Problem 1.** $ABCD$ is a square (see Figure 1), $F$ and $E$ are the points on the sides $AB$ and $DA$, respectively, such that $CEF$ is an equilateral triangle, $G$ and $H$ are points on the segment $EF$ such that $AGH$ is an equilateral triangle. Prove or disprove that the diameter of the incircle of $CEF$ equals $AG$.

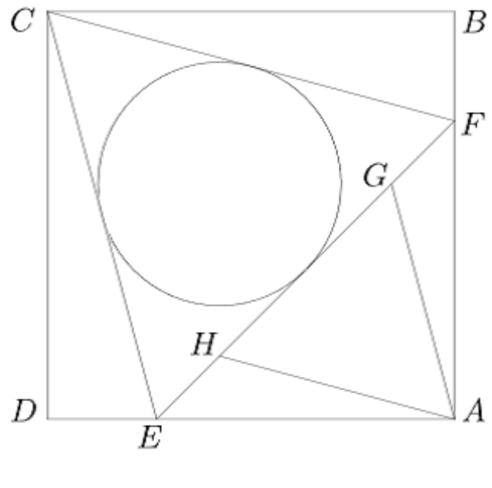


Figure 1.

**Solution.** Denote $I$ the foot of perpendicular from $G$ to $AB$, and $\mathcal{K}$ the incircle of $CEF$. Set $\theta = \angle BCF = \angle IAG$. Notice that $IF = IG$. Then

$$CB = AI + IF + FB$$

implies

$$CF \cos\theta = AG\cos\theta + AG\sin\theta + CF\sin\theta,$$

and consequently

$$\frac{AG}{CF} = \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = \frac{\cos 2\theta}{1 + \sin 2\theta} = \frac{\sqrt{3}/2}{1 + 1/2} = \frac{\sqrt{3}}{3}.$$

Finally, if $d$ is a diameter of $\mathcal{K}$, then

$$d = \frac{2}{3}\frac{\sqrt{3}}{2} CF = AG.$$

**Remark.** The equilateral triangle $AGH$ is homothetic to $CEF$ through a homothety $\mathscr{H}$ with center in the common midpoint $M$ of the segments $EF$ and $GH$, and ratio $-AG/CF$.

Applying $\mathscr{H}, \mathscr{H}^2, \ldots$ to the square $ABCD$, the triangle $CEF$ and the circle $\mathcal{K}$, we obtain a sequence of squares, equilateral triangles and their incircles alternating on both sides of $EF$ (see Figure 2).

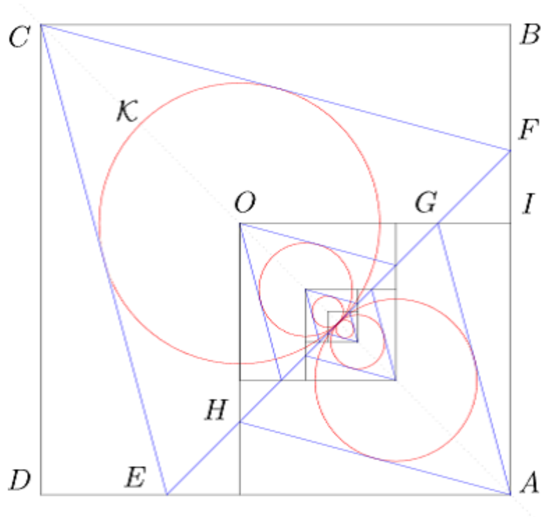


FIGURE 2.

Let $O$ be the center of $\mathcal{K}$. Notice that for $n = 0, 1, 2, \ldots$, $\mathscr{H}^n(O)$ coincides with $\mathscr{H}^{n+2}(C)$. To prove this, it suffices to verify $AO = AA'$, where $A' = \mathscr{H}(A) = \mathscr{H}^2(C)$.

From the squares, we obtain

$$AA' = \frac{\sqrt{3}}{3} CA = \frac{\sqrt{3}}{3}(CM + MA) = \frac{\sqrt{3}+1}{3} CM.$$

Since $C, M, A, O$ are collinear and $MO$ is inradius of $\triangle CEF$,

$$AO = AM + MO = \frac{\sqrt{3}+1}{3} CM.$$