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Integer Sequences and Circle Chains Inside a Tangential Quadrilateral

Giovanni Lucca
Via Corvi 20, 29122 Piacenza, Italy
e-mail: vanni_lucca@inwind.it

Abstract. By considering a generic tangential quadrilateral and its incircle, we study the four infinite chains of circles originating from the incircle itself and converging to each one of the quadrilateral vertices. Main property of the chains is that the generic i-th circle is tangent to the (i-1)-th and (i+1)-th ones and to two sides of the quadrilateral. Aim of the paper is looking for the conditions which guarantee that the ratio between the incircle radius and the generic i-th circle radius for each one of the chains is an integer number.

Keywords. Circle chains, Tangential quadrilaterals, Integer Sequences.

Mathematics Subject Classification (2010). 51M04.

1. Introduction

Let us consider a tangential quadrilateral i.e. a convex quadrilateral whose sides are all tangent to an inner circle said *incircle* within the quadrilateral itself.
In Fig.1 a generic tangential quadrilateral is shown and four circle chains originating from the common major circle (the incircle) are drawn inside it; each circle of the chain is tangent to the preceding and succeeding ones and to the two sides of the quadrilateral that delimit the chain itself.

It is possible to associate to each circle chain a sequence whose elements are given by the ratios of the incircle radius with the generic k-th circle radius; thus, if α, β,

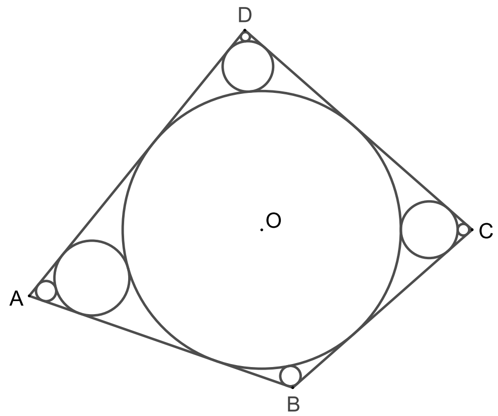

FIGURE 1. A General tangential quadrilateral with circle chains inscribed in it.

γ, δ are the angles corresponding to vertices A, B, C, D respectively, one obtains the following sequences:

$$\begin{aligned} &\{\tau_{\alpha k}\} \quad with \quad \tau_{\alpha k} = \frac{r_0}{r_{\alpha k}} \quad k \geq 0 \\ &\{\tau_{\beta k}\} \quad with \quad \tau_{\beta k} = \frac{r_0}{r_{\beta k}} \quad k \geq 0 \\ &\{\tau_{\gamma k}\} \quad with \quad \tau_{\gamma k} = \frac{r_0}{r_{\gamma k}} \quad k \geq 0 \\ &\{\tau_{\delta k}\} \quad with \quad \tau_{\delta k} = \frac{r_0}{r_{\delta k}} \quad k \geq 0 \end{aligned} \tag{1}$$

Being r_0 the incircle radius.
One may pose the following problem: *Determine the values of the angles α, β, γ, δ so that all the elements belonging to the four sequences defined by (1) are integer.*
The analogous question, raised for a generic triangle, and the relevant solution can be found in [1]; so, part of the material in it can be used also here in order to find the solution relevant to the above posed problem.

2. PRELIMINARIES

First of all it is necessary to report from [1], without proof, the following theorem:

Theorem 2.1. *Let us consider a circle chain in between two intersecting lines forming an angle θ. Then, the radii of the circle chain form a geometric progression of common ratio* $\frac{1+\sin\left(\frac{\theta}{2}\right)}{1-\sin\left(\frac{\theta}{2}\right)}$

A circle chain between two straight lines is shown in fig.2
On the basis of this theorem one can write that the radius r_i of the i-th circle and the radius r_{i-1} of the (i-1)-th circle are related by:

$$r_i = \left(\frac{1 - \sin\frac{\theta}{2}}{1 + \sin\frac{\theta}{2}} \right) r_{i-1} \tag{2}$$

Moreover, it is known that, in a convex quadrilateral, the angles α, β, γ, δ must satisfy the following conditions:

$$0<\alpha<\pi \quad\Rightarrow\quad \sin\left(\frac{\alpha}{2}\right)>0$$
$$0<\beta<\pi \quad\Rightarrow\quad \sin\left(\frac{\beta}{2}\right)>0$$
$$0<\gamma<\pi \quad\Rightarrow\quad \sin\left(\frac{\gamma}{2}\right)>0$$
$$\delta=2\pi-(\alpha+\beta+\gamma) \quad\Rightarrow\quad \sin\left(\frac{\alpha+\beta+\gamma}{2}\right)>0$$

Therefore, by taking into account Theorem 2.1 and formula (2) the sequences in (1) can be written as follows:

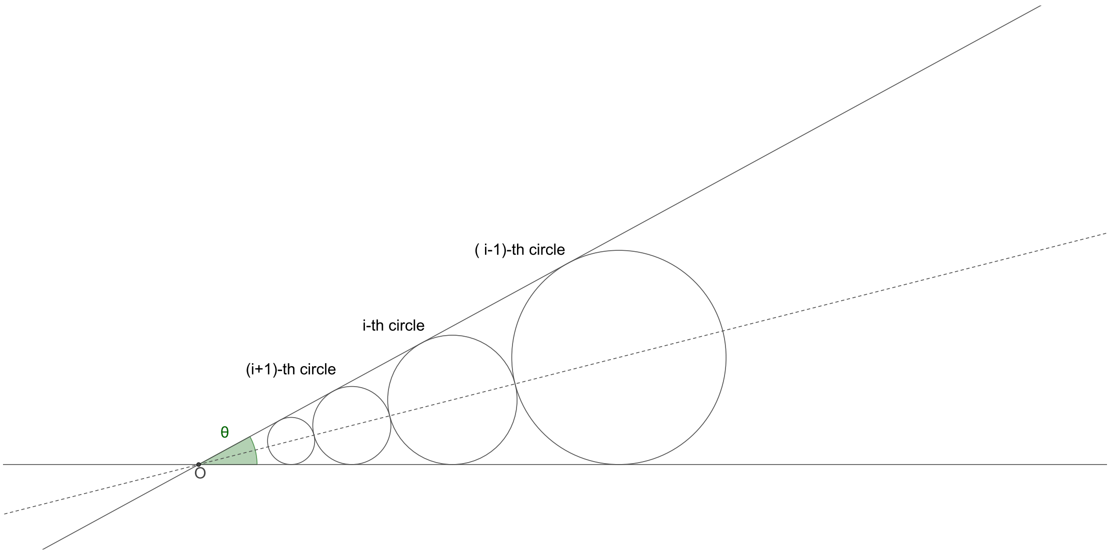

FIGURE 2. Circle chain between two straight lines.

(3)
$$\tau_{\alpha k}=\left(\frac{1+\sin\left(\frac{\alpha}{2}\right)}{1-\sin\left(\frac{\alpha}{2}\right)}\right)^k \quad k\geq 0$$

(4)
$$\tau_{\beta k}=\left(\frac{1+\sin\left(\frac{\beta}{2}\right)}{1-\sin\left(\frac{\beta}{2}\right)}\right)^k \quad k\geq 0$$

(5)
$$\tau_{\gamma k}=\left(\frac{1+\sin\left(\frac{\gamma}{2}\right)}{1-\sin\left(\frac{\gamma}{2}\right)}\right)^k \quad k\geq 0$$

(6)
$$\tau_{\delta k}=\left(\frac{1+\sin\left(\frac{\alpha+\beta+\gamma}{2}\right)}{1-\sin\left(\frac{\alpha+\beta+\gamma}{2}\right)}\right)^k \quad k\geq 0$$

3. Conditions to obtain integer sequences

As far as Eqns.(3)-(5) are concerned, in order that $\{\tau_{\alpha k}\}$, $\{\tau_{\beta k}\}$, $\{\tau_{\gamma k}\}$ are composed by integer numbers (respectively defined by j, m and n) it must be:

$$\alpha = 2\arcsin\frac{j-1}{j+1} \qquad j \geq 2 \tag{7}$$

$$\beta = 2\arcsin\frac{m-1}{m+1} \qquad m \geq 2 \tag{8}$$

$$\gamma = 2\arcsin\frac{n-1}{n+1} \qquad n \geq 2 \tag{9}$$

As far as equation (6) is concerned, one can see that $\tau_{\delta 0} = 1$ and, in order to be $\tau_{\delta 1}$ an integer, the following condition must hold:

$$\frac{1+\sin\left(\frac{\alpha+\beta+\gamma}{2}\right)}{1-\sin\left(\frac{\alpha+\beta+\gamma}{2}\right)} \geq 2 \tag{10}$$

(Infact, being: $\tau_{\delta 1} > \tau_{\delta 0} = \frac{r_0}{r_0} = 1$ and $\tau_{\delta k} \in \mathbb{Z}^+$, that implies $\tau_{\delta 1} \geq 2$). Therefore, from (10), one has:

$$\sin\left(\frac{\alpha+\beta+\gamma}{2}\right) \geq \frac{1}{3} \tag{11}$$

Moreover, in order to avoid division by 0 in (6), also the following condition must be satisfied:

$$\sin\left(\frac{\alpha+\beta+\gamma}{2}\right) \neq 1 \tag{12}$$

From (7), (8) and (9) one can obtain:

$$\sin\frac{\alpha}{2} = \frac{j-1}{j+1} \qquad \cos\frac{\alpha}{2} = \frac{2\sqrt{j}}{j+1} \tag{13}$$

$$\sin\frac{\beta}{2} = \frac{m-1}{m+1} \qquad \cos\frac{\beta}{2} = \frac{2\sqrt{m}}{m+1} \tag{14}$$

$$\sin\frac{\gamma}{2} = \frac{n-1}{n+1} \qquad \cos\frac{\gamma}{2} = \frac{2\sqrt{n}}{n+1} \tag{15}$$

Furthermore, one has the following identity:

$$\sin\left(\frac{\alpha+\beta+\gamma}{2}\right) = \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2} + \cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2} \tag{16}$$

By substituting (13), (14) and (15) into (6) and by taking into account of equation (16), through some algebraical steps one finally obtains:

$$\tau_{\delta k} = \left(\frac{jm+jn+mn+1+2(j-1)\sqrt{mn}+2(m-1)\sqrt{nj}+2(n-1)\sqrt{jm}}{jmn+j+m+n-2(j-1)\sqrt{mn}-2(m-1)\sqrt{nj}-2(n-1)\sqrt{jm}}\right)^k \quad k \geq 0$$

Both numerator and denominator of the above expression are squares so it can be simplified into:

$$\tau_{\delta k}=\left(\frac{\sqrt{jm}+\sqrt{mn}+\sqrt{nj}-1}{\sqrt{jmn}-\sqrt{j}-\sqrt{m}-\sqrt{n}}\right)^{2k}\qquad k\geq 0 \tag{17}$$

By substituting in (16) the expressions given in (13), (14) and (15) and taking into account the inequalities in (11) and (12), one has:

$$\frac{4\,(j-1)\sqrt{mn}+4\,(m-1)\sqrt{nj}+4\,(n-1)\sqrt{jm}-(j-1)\,(m-1)\,(n-1)}{(j+1)\,(m+1)\,(n+1)}\geq\frac{1}{3} \tag{18}$$

$$\frac{4\,(j-1)\sqrt{mn}+4\,(m-1)\sqrt{nj}+4\,(n-1)\sqrt{jm}-(j-1)\,(m-1)\,(n-1)}{(j+1)\,(m+1)\,(n+1)}\neq 1 \tag{19}$$

Thus, one has to find all the triplets of integers (j, m, n) that, subjected to conditions (18) and (19), generate from (17) the sequence $\{\tau_{\delta k}\}$ composed by only integers.
By looking at formulas (17), (18) and (19), one can notice that by substituting in them any permutation of the triplet (j, m, n) one obtains the same result. For example, the triplet (3, 4, 5) generates the same result as the triplets $(4, 3, 5)$ or $(3, 5, 4)$. This remark is important for the following because it helps in finding the triplets that yield integer sequences when substituted inside (17).

4. Triplets generating integer sequences

The following theorem holds:

Theorem 4.1. *The only triplets yielding integer sequences from equation (17) and satisfying the conditions (18) and (19) are* $(4, 4, 9)$ *and* $(4, 9, 9)$ *with all their permutations.*

Proof. Firstly, let us define, for convenience, the following function:

$$f(j,m,n)=\left(\frac{\sqrt{jm}+\sqrt{mn}+\sqrt{nj}-1}{\sqrt{jmn}-\sqrt{j}-\sqrt{m}-\sqrt{n}}\right)^{2} \tag{20}$$

The rationale of the proof is based on the following main ideas that introduce successive levels of selection of the triplets (j, m, n) :

(1) FIRST LEVEL OF SELECTION. To consider only the triplets (j, m, n) yielding a rational number when substituted inside formula (20). Thus, all the triplets generating an irrational number must be *a priori* excluded. So, by looking at (20), the only triplets (j, m, n) allowed (that is the ones generating a rational number) are of two kinds:

triplets of kind 1: the triplet (j, m, n) is composed by square integer numbers multiplied by a common integer factor p that is:
$(p\cdot q_1^2, p\cdot q_2^2, p\cdot q_3^2)$

triplets of kind 2: the triplet (j, m, n) is composed by square integer numbers (q_1^2, q_2^2, q_3^2)

(2) SECOND LEVEL OF SELECTION. To exclude all the triplets (j, m, n) not fullfilling conditions given by (18) and (19). In particular, by looking at the ratio in formula (18), one can notice that, it decreases by increasing the value of the integers j, m and n. This greatly helps in identifying the triplets that do not fullfill inequality (18). Infact, when one finds the triplet $(j_{min}, m_{min}, n_{min})$ having the minimum values for j, m and n not fullfilling inequality (18), one also has that all the other triplets with larger values of for j, m and n do not satisfy (18) as well.

(3) THIRD LEVEL OF SELECTION. If a triplet (j, m, n) has passed the first two levels of selection, the last step is to verify, by substitution in (20), if it generates an integer number.

We shall consider separately three cases: CASE 1 with $j = m = n$, CASE 2 with $j \neq m \quad m \neq n \quad n \neq j$ and CASE 3 with $j = m \quad m \neq n$. In CASE 1, each

triplet (j, j, j) always generates a rational number when substituted in (20) but conditions (18) and (19) are fullfilled only by the following finite set of values for j: $\{2, 4, 5, 6, 7, 8, 9\}$. By direct substitution of these values inside (20), the function $f(j, j, j)$ never assumes integer values, so neither does sequence $\{\tau_{\delta k}\}$.

In CASE 2, the triplets generating a rational number when substituted in (20) and fulfilling conditions (18) and (19) are of the following three types:

type 1 : infinite set composed by triplets of kind 1 (having a common factor 2) of the type: $\left\{2 \cdot 1^2, 2 \cdot \left(2^{k_1}\right)^2, 2 \cdot \left(2^{k_2}\right)^2\right\}$ being $k_1 = 1$ and $k_2 \geq 2$.

By substitution inside (20), the numerator is always an odd number while the denominator is always an even number, so their ratio cannot be an integer.

type 2 : finite set composed by the only triplet of kind 1 (having common factor 3):
$\{(3 \cdot 1^2, 3 \cdot 2^2, 3 \cdot 3^2)\}$

In this case, direct substitution of the triplet inside (20) yields a non integer number.

type 3 : finite set composed by the two following triplets of kind 2 formed by square numbers: $\{(2^2, 3^2, 4^2), (2^2, 3^2, 5^2)\}$.

By direct substitution of these values inside (20), the function $f(j, m, n)$ never assumes integer values

Any other triplet of kind 1 or 2, obtained by increasing the values of j, m, n, generates a rational number by (20) but does not fullfill condition (18). Hence, one can conclude that also in CASE 2, the function $f(j, m, n)$ never assumes integer values, so neither does sequence $\{\tau_{\delta k}\}$.

In CASE 3 the triplets generating a rational number when substituted in (20) and fulfilling conditions (18) and (19) are of the following five types:

type 1 : finite set composed by the two triplets of kind 1 (having common factor 2) of the type: $\left\{2 \cdot 1^2, 2 \cdot \left(2^{l_1}\right)^2, 2 \cdot \left(2^{l_1}\right)^2\right\}$ being $l_1 = 1, 2$

By substitution inside (20), the numerator is always an odd number while the denominator is always an even number, so their ratio cannot be an integer. For $l_1 > 2$ equation (18) is not fullfilled.

type 2 : infinite set composed by the triplets of kind 1 (having common factor 2) of the type: $\left\{2 \cdot 1^2, 2 \cdot 1^2, 2 \cdot \left(2^{l_2}\right)^2\right\}$ being l_2 an integer ≥ 2

By substitution inside (20), the numerator is always an odd number while the denominator is always an even number, so their ratio cannot be an integer.

type 3 : infinite set composed by the triplets of kind 1 (having common factor 3): $\left\{\left(3 \cdot 1^2, 3 \cdot 1^2, 3 \cdot \left(2^{l_3}\right)^2\right)\right\}$ being l_3 an integer ≥ 1

In this case, direct substitution of the triplets inside (20) yields an expression of the type.

$$\left[\frac{3 \cdot 2^{l_3} + 1}{2^{l_3} - 1}\right]^2 \frac{1}{3} = \left[3 + \frac{4}{2^{l_3} - 1}\right]^2 \frac{1}{3}$$

where one can notice that the expression inside square parentheses can never be a multiple of 3 and so the whole expression cannot be an integer.

type 4 : infinite set composed by the triplets of kind 1 (having common factor 3): $\left\{\left(3 \cdot 1^2, 3 \cdot 1^2, 3 \cdot \left(3^{l_4}\right)^2\right)\right\}$ being l_4 an integer ≥ 1

In this case, direct substitution of the triplets inside (20) yields an expression of the type.

$$\left[\frac{3^{l_4+1} + 1}{3^{l_4} - 1}\right]^2 \frac{1}{3} = \left[3 + \frac{4}{3^{l_4} - 1}\right]^2 \frac{1}{3}$$

where one can notice that the expression inside square parentheses can never be a multiple of 3 and so the whole expression cannot be an integer.

type 5 : finite set composed by triplets of kind 2 formed by square numbers as: $(2^2, 2^2, l_5^2)$ with $l_5 = 3, 4, ..., 35$ and by the triplet $(3^2, 3^2, 2^2)$.

By direct substitution inside (20) one gets that the only triplets yielding an integer value are: $\{(4, 4, 9), (9, 9, 4)\}$ so obtaining from (20):

$$f(4, 4, 9) = 9 \qquad f(9, 9, 4) = 4 \tag{21}$$

Any other triplet of kind 1 or 2, obtained by increasing the values of j, m, n, generates a rational number by (20) but does not fullfill condition (18). This concludes the proof. □

5. Quadrilaterals compatible with integer sequences

The triplet $(4, 4, 9)$ and its permutations correspond to the angle $\delta = 2 \arcsin\left(\frac{4}{5}\right)$ while the triplet $(4, 9, 9)$ and its permutations correspond to the angle $\delta = 2 \arcsin\left(\frac{3}{5}\right)$. In fact, one has in analogy with (7), (8), (9):

$$\delta = 2 \arcsin \frac{f(j,j,n) - 1}{f(j,j,n) + 1} \tag{22}$$

On the basis of formulas (21) and (22) one can state:

Theorem 5.1. *It is possible to inscribe inside a tangential quadrilateral four circle chains generating integer sequences*

$$\begin{aligned} \{\tau_{\alpha k}\} \quad &\textit{with} \quad \tau_{\alpha k} = \frac{r_0}{r_{\alpha k}} \in \mathbb{Z}^+ \\ \{\tau_{\beta k}\} \quad &\textit{with} \quad \tau_{\beta k} = \frac{r_0}{r_{\beta k}} \in \mathbb{Z}^+ \\ \{\tau_{\gamma k}\} \quad &\textit{with} \quad \tau_{\gamma k} = \frac{r_0}{r_{\gamma k}} \in \mathbb{Z}^+ \\ \{\tau_{\delta k}\} \quad &\textit{with} \quad \tau_{\delta k} = \frac{r_0}{r_{\delta k}} \in \mathbb{Z}^+ \end{aligned}$$

if and only if the quadrilateral is a rhombus or an isosceles trapezoid[2] *having angles given by:* $\arcsin\left(\frac{3}{5}\right)$ *and* $\arcsin\left(\frac{4}{5}\right)$.

Proof. By taking into account the formulas (7), (8), (9) and (22), a biunivocal correspondence is established between the quadruplet of integers $(j, m, n, f(j, m, n))$ and the quadruplet of angles $(\alpha, \beta, \gamma, \delta)$. Moreover, the quadruplets are formed only by a pair of 4's and pair of 9's. If two equal components of the quadruplet occupy consecutive positions inside the quadruplet itself, this means that the corresponding equal angles have a common side and consequently the quadrilateral is an isosceles trapezoid. On the contrary, if the equal components of the quadruplet occupy alternate positions, the corresponding equal angles are opposite and the quadrilateral is a rhombus. □

Finally, one can build up the following table:
The integer sequences generated by the circle chains inscribed inside the quadrilateral are then: $\{4^k\}$ and $\{9^k\}$; they are classified in OEIS (On Line Encyclopedia of Integer Sequences) [2] as: A000302 and A001019 respectively. In Fig.3 an example of a rhombus with four inscribed circle chains is shown. In Fig.4 an example of a trapezoid with four inscribed circle chains is shown. It may be interesting to add a further remark concerning a property of both the rhombus and trapezoid shown in Figs.3 and 4. Such a property relates them to the so called *3-4-5 triangle* whose sides form the basic Pythagorean triplet $(3, 4, 5)$.
In fact, if we draw the segments connecting the incentre of the quadrilateral to each of the vertices and to each of the tangency points of the incircle with the

[2] isosceles trapezium in British English

integer quadruplets	angles	quadrilateral
$[j, m, n, f(j,m,n)]$	$(\alpha, \beta, \gamma, \delta)$	———
$(4,4,9,9)$	$(\arcsin(3/5), \arcsin(3/5), \arcsin(4/5), \arcsin(4/5))$	isosceles trapezoid
$(4,9,4,9)$	$(\arcsin(3/5), \arcsin(4/5), \arcsin(3/5), \arcsin(4/5))$	rhombus
$(9,4,4,9)$	$(\arcsin(4/5), \arcsin(3/5), \arcsin(3/5), \arcsin(4/5))$	isosceles trapezoid
$(4,9,9,4)$	$(\arcsin(3/5), \arcsin(4/5), \arcsin(4/5), \arcsin(3/5))$	isosceles trapezoid
$(9,4,9,4)$	$(\arcsin(4/5), \arcsin(3/5), \arcsin(4/5), \arcsin(3/5))$	rhombus
$(9,9,4,4)$	$(\arcsin(4/5), \arcsin(4/5), \arcsin(3/5), \arcsin(3/5))$	isosceles trapezoid

TABLE 1. Correspondance between integer quadruplets, angles and types of quadrilaterals.

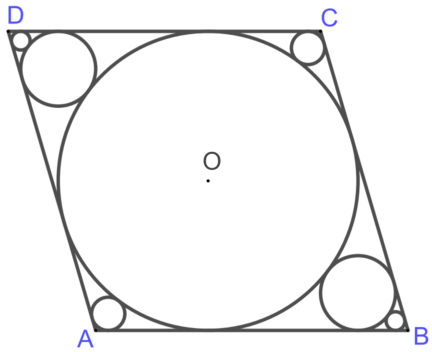

FIGURE 3. Rhombus with circle chains inscribed in it

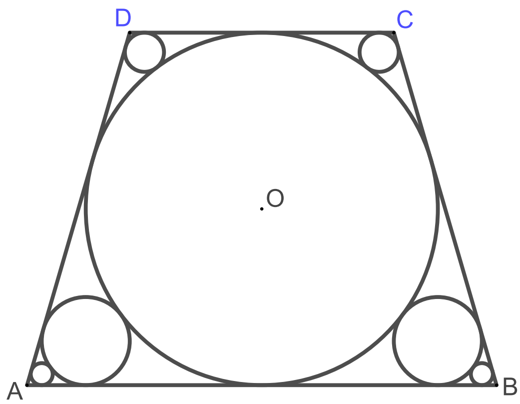

FIGURE 4. Trapezoid with circle chains inscribed in it

quadrilateral sides, one has that the quadrilateral itself is subdivided into eight right triangles having angles equal to $\arcsin\left(\frac{3}{5}\right)$ and $\arcsin\left(\frac{4}{5}\right)$ so meaning that each one of them is similar to the 3-4-5 triangle.
Therefore, the above mentioned property is given by the fact that both the rhombus and trapezoid can be seen as the union of eight triangles all similar to the 3-4-5 triangle. See Fig.5.

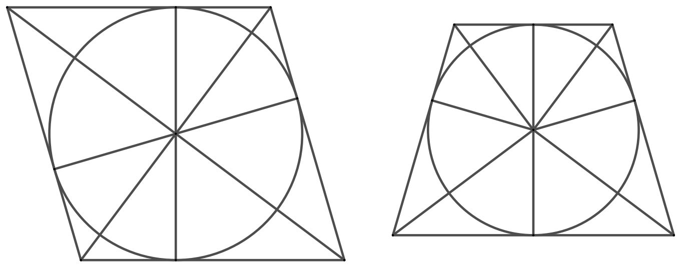

FIGURE 5. Rhombus and trapezoid composed by *3-4-5 triangles*

6. APPENDIX

It is possible to deduce formula (17) in a different way; in fact, in paper [3] it is reported a formula relating the tangential quadrilateral incircle radius r_0 with the radii of the first circles of the four chains corresponding to the angles α, β, γ, δ, that is: $r_{1\alpha}$, $r_{1\beta}$, $r_{1\gamma}$, $r_{1\delta}$. The formula is:

$$r_0^2 - \left(\sqrt{r_{1\alpha}r_{1\beta}} + \sqrt{r_{1\alpha}r_{1\gamma}} + \sqrt{r_{1\alpha}r_{1\delta}} + \sqrt{r_{1\beta}r_{1\gamma}} + \sqrt{r_{1\beta}r_{1\delta}} + \sqrt{r_{1\gamma}r_{1\delta}}\right) r_0 + \sqrt{r_{1\alpha}r_{1\beta}r_{1\gamma}r_{1\delta}} = 0$$

that can be rewritten as:

$$1 - \left(\sqrt{\frac{r_{1\alpha}}{r_0}\frac{r_{1\beta}}{r_0}} + \sqrt{\frac{r_{1\alpha}}{r_0}\frac{r_{1\gamma}}{r_0}} + \sqrt{\frac{r_{1\alpha}}{r_0}\frac{r_{1\delta}}{r_0}} + \sqrt{\frac{r_{1\beta}}{r_0}\frac{r_{1\gamma}}{r_0}} + \sqrt{\frac{r_{1\beta}}{r_0}\frac{r_{1\delta}}{r_0}} + \sqrt{\frac{r_{1\gamma}}{r_0}\frac{r_{1\delta}}{r_0}}\right) + \sqrt{\frac{r_{1\alpha}}{r_0}\frac{r_{1\beta}}{r_0}\frac{r_{1\gamma}}{r_0}\frac{r_{1\delta}}{r_0}} = 0$$

By imposing that $r_{1\alpha}/r_0 = 1/j$, $r_{1\beta}/r_0 = 1/m$, $r_{1\gamma}/r_0 = 1/n$, and by introducing the auxiliary variable $t = r_{1\delta}/r_0$, one obtains:

$$t\left(\frac{1}{\sqrt{j}} + \frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{jmn}}\right) = 1 - \frac{1}{\sqrt{jm}} - \frac{1}{\sqrt{mn}} - \frac{1}{\sqrt{jn}}$$

yielding the solution:

$$t = \frac{\sqrt{jmn} - \sqrt{j} - \sqrt{m} - \sqrt{n}}{\sqrt{jm} + \sqrt{mn} + \sqrt{jn} - 1}$$

From it, one immediately obtains:

$$\frac{r_0}{r_{1\delta}} = \tau_{\delta 1} = \left(\frac{\sqrt{jm} + \sqrt{mn} + \sqrt{jn} - 1}{\sqrt{jmn} - \sqrt{j} - \sqrt{m} - \sqrt{n}}\right)^2$$

The elements $\tau_{\delta k}$ of the sequence are simply the k-th power of the expression in the above expression.

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