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An Unsolved Problem in the Yamaguchi's Travell Diary

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Abstract. Yamaguchi Kanzan (1781?-1850) travelled Japan from 1817 to 1828 and recorded about 350 sangaku problems in his travell diary and some of his collection are unsolved and two of them are introduced by [2] Fukagawa and Rothman 2008. This paper gives an solution for the problem proposed by Sawa.

Keywords. Sawa's unsolved problem

Mathematics Subject Classification (2010). 01A20, 51M04

1. Sawa Problem

An ellipse is inscribed in a right triangle (see Figure 1), and the major axis of the ellipse is parallel to the hypotenuse of the triangle. Two circles are inscribed in the ellipse, and another circle is contact with both the ellipse and the triangle, and the radii of these three circles are equal. Determine the radius in terms of the sides of the triangle.

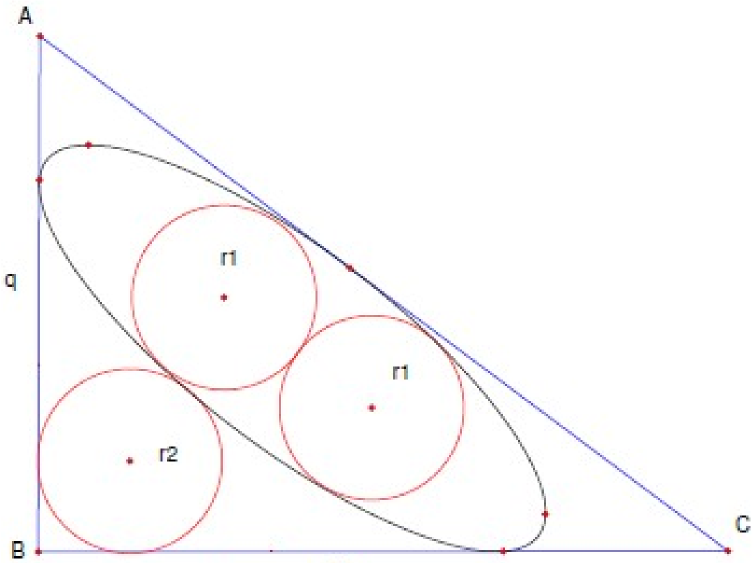

Figure 1: Sawa Problem, $\angle BCA = \delta, AC = 1$.

2. Solution

Let a^* and b^* be a semi-major axis and a semi-minor axis, and (m, n) be the center of the ellipse, and p and q be the sides of a right triangle. We choose the major axis as x^* axis, which is parallel to the side AC, and the minor-axis as y^* axis and (m, n) as the origin of the coordinate. Let BC be x axis and BA be y axis. The ellipse is given in (x^*, y^*) coordinate as

$$\frac{x^{*2}}{a^{*2}} + \frac{y^{*2}}{b^{*2}} = 1.$$

The transformation between (x, y) and (x^*, y^*) is

$$x^* = (x - m)\cos\delta - (y - n)\sin\delta, y^* = (x - m)\sin\delta + (y - n)\cos\delta$$

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Here we choose the side CA as an unit of length and $\angle BCA = \delta$, and, therefore, $\cos\delta = p, \sin\delta = q$. The equation of the ellipse takes the following form in (x, y) coordinate:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

(1) $$a = p^2/a^{*2} + q^2/b^{*2},$$

(2) $$b = q^2/a^{*2} + p^2/b^{*2},$$

(3) $$h = (1/b^{*2} - 1/a^{*2})pq,$$

(4) $$g = -m(p^2/a^{*2} + q^2/b^{*2}) + n(1/a^{*2} - 1/b^{*2})pq = -ma - nh,$$

(5) $$f = m(1/a^{*2} - 1/b^{*2})pq - n(q^2/a^{*2} + p^2/b^{*2}) = -nb - mh,$$

(6) $$c = (-mp + nq)^2/a^{*2} + (mq + np)^2/b^{*2} - 1 = m^2a + n^2b + 2mnh - 1.$$

Since the ellipse is tangent to the x axis, the equation $ax^2+2gx+c=0$ has equal roots:

$$g^2-ac=0.$$

Substituting equations(1),(4),and (6) into this equation, we have

$$n=\sqrt{a^{*2}q^2+b^{*2}p^2}. \tag{7}$$

From the condition that the ellipse is tangents to the y axis, we have similarly get

$$m=\sqrt{a^{*2}p^2+b^{*2}q^2}. \tag{8}$$

The equation of a line CA is

$$qx+py-pq=0. \tag{9}$$

The distance from the center of the ellipse(m,n) to the line(9)is $pq-qm-pn$ and is equal to the minor-axis b^*, then we have

$$b^*=pq-qm-pn. \tag{10}$$

From Lemma 1(see Appendix) the radius r_1 of the circle inscribed in the ellipse

$$r_1=\frac{b^*}{a^*}\sqrt{a^{*2}-b^{*2}} \tag{11}$$

From Ajima's theorem(see Appendix) the radius r_2 of the circle, which contacts with both the right triangle and the ellipse, is

$$r_2=s-\sqrt{s^2-h}, \tag{12}$$

where $s=m+n+\sqrt{k}$, and

$$h=mn(1+\alpha),k=mn(1-\alpha),\alpha=\sqrt{1-(a^*b^*/mn)^2}, \tag{13}$$

and we have from the condition of the problem:

$$r_1=r_2. \tag{14}$$

The equation(10), in which m,n are replaced by (7) and (8), is a function of unknown parameters a^*,b^*. The equation (14) is also a function of unknown parameters a^*,b^*. Therefore with use of (10) and (14) we can determine a^*,b^* and then can determine the radius of the circle.

3. An explicit equation for b^*

The equation (10) is expressed with use of (7) and (8):

$$b^*=pq-q\sqrt{a^{*2}p^2+b^{*2}q^2}-p\sqrt{a^{*2}q^2+b^{*2}p^2},$$

from which a^* is derived as

$$a^*=\frac{pq(pq-2b^*)(pq-b^*-b^*(p^2-q^2))(pq-b^*+b^*(p^2-q^2))}{2(pq-b^*)^2}. \tag{15}$$

Substituting (15) into (11),(7), and (8) we have

$$r_1=b^*\sqrt{\frac{pq(pq-2b^*-2b^{*2})(pq-2b^*+2b^{*2})}{(pq-2b^*)(pq-b^*-b^*(p^2-q^2))(pq-b^*+b^*(p^2-q^2))}},$$

$$m=\frac{(pq-b^*)^2-b^{*2}(p^2-q^2)}{2q(pq-b^*)}, \tag{16}$$

$$n = \frac{(pq - b^*)^2 + b^{*2}(p^2 - q^2)}{2p(pq - b^*)}. \tag{17}$$

With use of (15),(16), and (17), we get

$$\sqrt{(mn)^2 - (a^*b^*)^2} = \frac{pq(-b^{*4} + b^{*2} - b^*pq + p^2q^2/4)}{(pq - b^*)^2}.$$

Then from (13) we obtain

$$h = mn + \sqrt{(mn)^2 - (a^*b^*)^2} = \frac{pq}{2} - b^*, \tag{18}$$

$$k = mn - \sqrt{(mn)^2 - (a^*b^*)^2} = \frac{b^{*2}(4pqb^{*2} - 2b^* + pq)}{2(pq - b^*)^2}. \tag{19}$$

By substituting (16),(17),(18), and (19) into (12), r_2 becomes a function of b^* only, and also r_1 is a function of b^*only. Now the relation (14) is an equation for b^*. Rationalizing the equation (14) by squaring three times, we obtain an explicit equation for b^*. The degree of of this polynomial with respect to b^* is 28 and the number terms is 2455. With use of $p^2 + q^2 = 1$ the number of terms is reduced to 465, whose expression is still quite long and is not presented here.

4. CASE FOR AN ISOSCELES TRIANGLE ($p = q = 1/\sqrt{2}$)

From (7),(8), and then (13) we have

$$m = n = \sqrt{(a^{*2} + b^{*2})/2}, \alpha = (a^{*2} - b^{*2})/(a^{*2} + b^{*2}),$$

and get from(12)

$$r_2 = (\sqrt{2} - 1)(\sqrt{a^{*2} + b^{*2}} - b^*), \tag{20}$$

The condition (10) takes the following form:

$$\sqrt{a^{*2} + b^{*2}} = \frac{1}{2} - b^*,$$

from which we obtain

$$a^{*2} = \frac{1}{4} - b^*. \tag{21}$$

Now substituting (21) into (14) with (11) and (20), we get an explicit equation for b^*:

$$16b^{*4} + 16(-11 + 8\sqrt{2})b^{*3} + 4(35 - 24\sqrt{2})b^{*2} + 12(-3 + 2\sqrt{2})b^* + 3 - 2\sqrt{2} = 0. \tag{22}$$

The solutions of (22) are $b^* = 0.116608, 0.206143$ and two complex roots. The value of a^*, which corresponds with the second real solution, is 0.209421,which means $a^* < 2\sqrt{b^*}$ and two equal circles cannot be inscribed in the ellipse. Therefore the solution is

$$a^* = 0.365228, b^* = 0.116608, r = 0.110505, m = n = 0.271099.$$

5. Numerical Examples

Example 1: $p = 4/5, q = 3/5$

The equation of b^* obtained from the polynomial mentioned above is

$$451377585192960 - 2633035913625600\, b^* + 70105835026907136\, b^{*2} - 1123222297514803200\, b^{*3} + 11962026434911113216\, b^{*4} - 87348823863955660800\, b^{*5} + 421893850123855742976\, b^{*6} - 1040694777054550886400\, b^{*7} - 223653598702189996262 4\, b^{*8} + 36661197341550571776000\, b^{*9} - 202196054235733212960000\, b^{*10} + 734525400838032576000000\, b^{*11} - 1972342803660832028250000\, b^{*12} + 4110067171731820837500000\, b^{*13} - 6915756241551655771875000\, b^{*14} + 9842804413718731171875000\, b^{*15} - 12253097167126170615234375\, b^{*16} + 12768870923260155273437500\, b^{*17} - 8941833305111340332031250\, b^{*18} + 4680814002197265625000000\, b^{*19} + 7012123832519073486328125\, b^{*20} - 6772677637283325195312500\, b^{*21} + 10178600687026977539062500\, b^{*22} + 15235650920867919921875000\, b^{*23} + 11993526792526245117187500\, b^{*24} - 33527796268463134765625000\, b^{*25} + 22431910037994384765625000\, b^{*26} - 56684017181396484375000000\, b^{*27} + 23283064365386962890625\, b^{*28}.$$

This equation has 6 real roots and 22 complex roots. Among the real roots, 4 roots are plus:

0.11264, 0.19911, 0.24294, 0.36041,

and corresponding a^* are

0.36289, 0.202273, 0.05299i, 0.19005i

Only the first set($a^* = 0.36289$ and $b^* = 0.11264$)satisfies $a^* > \sqrt{2}b^*$. The corresponding radius is 0.107076.

Figure 1 corresponds with this solution.

Example 2: an isosceles triangle($p = q = 1/\sqrt{2}$)

The equation for b^* is

(23)

$$1 - 56\, b^* + 1432\, b^{*2} - 22048\, b^{*3} + 225808\, b^{*4} - 1586944\, b^{*5} + 7380224\, b^{*6} - 17470464\, b^{*7} - 37689344\, b^{*8} + 588251136\, b^{*9} - 3149385728\, b^{*10} + 11160731648\, b^{*11} - 29357481984\, b^{*12} + 60146581504\, b^{*13} - 99649388544\, b^{*14} + 139183521792\, b^{*15} - 168678785024\, b^{*16} + 170090561536\, b^{*17} - 115262619648\, b^{*18} + 4936695808\, b^{*19} + 89229623296\, b^{*20} - 86704652288\, b^{*21} + 16559112192\, b^{*22} + 14965276672\, b^{*23} + 16928210944\, b^{*24} - 42010148864\, b^{*25} + 27783069696\, b^{*26} - 6979321856\, b^{*27} + 268435456\, b^{*28}.$$

This polynomial is factorized as

$$(2b^* - 1)^{12} P_1(b^*) * P_2(b^*) = 0,$$

$$P_1(b^*) = 1 - 24\, b^* + 216\, b^{*2} - 896\, b^{*3} + 1648\, b^{*4} - 1280\, b^{*5} + 2688\, b^{*6} - 5632\, b^{*7} + 256\, b^{*8},$$

$$P_2(b^*) = 1 - 8\, b^* - 8\, b^{*2} + 192\, b^{*3} - 272\, b^{*4} - 512\, b^{*5} + 128\, b^{*6} + 512\, b^{*7} + 256\, b^{*8}.$$

(22) is rearranged in the following form:

$$Q_1(b^*) + \sqrt{2} Q_2(b^*) = 0,$$

$$Q_1(b^*) = 16b^{*4} - 176b^{*3} + 140b^{*2} - 36b^{*2} + 3,$$

$$Q_2(b^*) = 128b^{*3} - 96b^{*2} + 24b^* - 2.$$

And then we get

$Q_1^2 - 2Q_2^2 = 1 - 24\,b^* + 216\,b^{*2} - 896\,b^{*3} + 1648\,b^{*4} - 1280\,b^{*5} + 2688\,b^{*6} - 5632\,b^{*7} + 256\,b^{*8}$.

This equation is equivalent with $P_1(b^*)$,which indicates that the equation(23) is correct.

$P_2(b^*) = 0$ has 4 real roots and 4 complex roots. Among 4 real roots, two roots are plus: 0.252938, 0.817934 and the corresponding a^* are 0.0542042i, -0.753614i,which do not satisfy $a^* > \sqrt{2}b^*$, therefore both are not solutions.

Appendix

Lemma 1. *Two circles with a same radius are inscribed in an ellipse with axes a and b. The radius of the circles is*

$$r = \frac{b}{a}\sqrt{a^2 - b^2}.$$

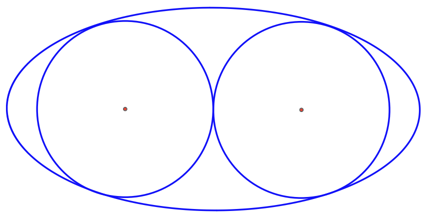

Figure 2: Lemma 1

Proof. The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{24}$$

and the equation of the circle is

$$(x - r)^2 + y^2 = r^2. \tag{25}$$

By substituting $y^2 = 2rx - x^2$ from (25) into (24), we have

$$\left(\frac{1}{a^2} - \frac{1}{b^2}\right)x^2 + \frac{2r}{b^2}x - 1 = 0. \tag{26}$$

The condition that the circle (25) is tangent to the ellipse (24) is that the equation (26) has equal roots.

$$\left(\frac{r}{b^2}\right)^2 + \frac{1}{a^2} - \frac{1}{b^2} = 0,$$

from this equation, we have

$$r = \frac{b}{a}\sqrt{a^2 - b^2}.$$

The x coordinate of the contact point,x_m, is obtained from (26):

$$x_m = a^2 r/(a^2 - b^2) = ab/\sqrt{a^2 - b^2}.$$

Since x_m should be less than a, then we have

$$a \geq \sqrt{2}b.$$

□

Lemma 2. *Consider two conic curves*

(27) $$k^2\left(\frac{x}{a}+\frac{y}{b}-1\right)^2-4xy=0.$$

(28) $$k'^2\left(\frac{x}{a'}+\frac{y}{b'}-1\right)^2-4xy=0,$$

which are tangent to x axis at a, a' and to y axis at b, b' . The condition that these two conic curves are tangent to each other is

$$k^2k'^2\left(\frac{1}{a}-\frac{1}{a'}\right)\left(\frac{1}{b}-\frac{1}{b'}\right)-(k\pm k')^2=0.$$

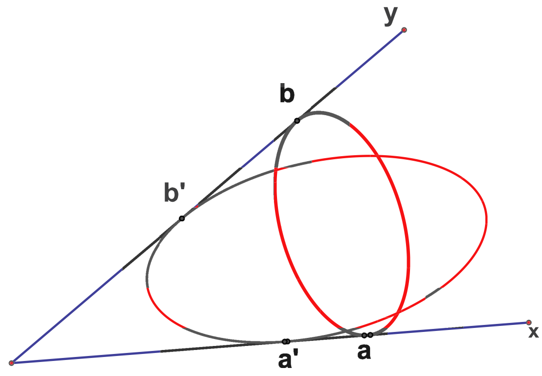

Figure 3: Lemma 2

Proof. This lemma can be applied to an oblique coordinate system.
The difference between (27)and (28) is
(29)
$$\left(\left(\frac{k}{a}+\frac{k'}{a'}\right)x+\left(\frac{k}{b}+\frac{k'}{b'}\right)y-(k+k')\right)\left(\left(\frac{k}{a}-\frac{k'}{a'}\right)x+\left(\frac{k}{b}-\frac{k'}{b'}\right)y-(k-k')\right)=0,$$

whose factors are the secant lines of the two conic curves. Consider the line obtained from the first factor of (29)

(30) $$\left(\frac{k}{a}+\frac{k'}{a'}\right)x+\left(\frac{k}{b}+\frac{k'}{b'}\right)y-(k+k')=0.$$

From this equation we have

(31) $$y=\left(k+k'-\left(\frac{k}{a}+\frac{k'}{a'}\right)x\right)\Big/\left(\frac{k}{b}+\frac{k'}{b'}\right)$$

Substituting (31) into the equation (27), we get a quadratic equation for x, which gives points of intersection:

(32) $$dx^2+2ex+f=0,$$

$$d=k^2k'^2\left(\frac{1}{ab'}-\frac{1}{ba'}\right)^2+4\left(\frac{k}{a}+\frac{k'}{a'}\right)\left(\frac{k}{b}+\frac{k'}{b'}\right),$$

$$e=k^2k'^2\left(\left(\frac{1}{b}-\frac{1}{b'}\right)\left(\frac{1}{ab'}-\frac{1}{ba'}\right)-2(k+k')\left(\frac{k}{b}+\frac{k'}{b'}\right)\right),$$

$$f=k^2k'^2\left(\frac{1}{b}-\frac{1}{b'}\right)^2.$$

The condition that the line (30) is tangent to the conics is that the discriminant of (32) is zero,

$$e^2 - df = 0.$$

From this we get

$$k^2k'^2\left(\frac{1}{a}-\frac{1}{a'}\right)\left(\frac{1}{b}-\frac{1}{b'}\right)-(k+k')^2=0.$$

Similarly from the second factor (29), we get the condition that this line is tangent to the conics,

$$k^2k'^2\left(\frac{1}{a}-\frac{1}{a'}\right)\left(\frac{1}{b}-\frac{1}{b'}\right)-(k-k')^2=0.$$

□

Lemma 3. *An ellipse with a semi-major axis a and a semi-minor axis b is inscribed in a rectangle whose sides are $2m$ and $2n$. The $x-y$ and x^*-y^* coordinates are introduced as Figure 4. The origin is the center of both the ellipse and the rectangle, and x axis is parallel to the side AB and the x^* axis is the major axis of the ellipse. The equation of the ellipse in x^*-y^* coordinate is*

(33) $$b^2x^{*2}+a^2y^{*2}=a^2b^2,$$

and the equation of the ellipse in $x-y$ coordinate is

$$n^2x^2 \pm 2\sqrt{m^2n^2-a^2b^2}xy+m^2y^2=a^2b^2.$$

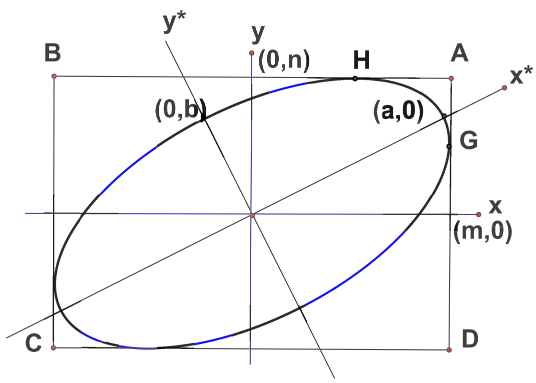

Figure 4: Lemma 3, The angle between x axis and x^* axis is ω.

Proof. A point on the ellipse is expressed as

$$x^*=a\cos u, y^*=b\sin u,$$

where u is an eccentric angle and which satisfies the equation(33). The coordinates of this point in $x-y$ with use of x^*, y^* are

(34) $$x=x^*\cos\omega-y^*\sin\omega, y=x^*\sin\omega+y^*\cos\omega.$$

The derivative of x with respect u is

$$\frac{dx}{du}=-a\sin u\cos\omega-b\cos u\sin\omega.$$

At the contact point G of the ellipse wit the rectangle, x takes a maximum value:

(35) $$a\sin u_G\cos\omega+b\cos u_G\sin\omega=0,$$

(36) $$a\cos u_g\cos\omega-\sin u_G\sin\omega=m.$$

From the contact condition at H, we get similarly

$$-a\sin u_H \sin\omega + b\cos u_H \cos\omega = 0, \tag{37}$$

$$a\cos u_H \sin\omega + b\sin u_H \cos\omega = n. \tag{38}$$

Eliminating u_G from(35) and (36), we have

$$a^2\cos^2\omega + b^2\sin^2\omega = m^2. \tag{39}$$

Similarly from (37) and (38) we have

$$a^2\sin^2\omega + b^2\cos^2\omega = n^2. \tag{40}$$

From (39) and(40) we get the following relations:

$$a^2 + b^2 = m^2 + n^2,$$

$$\sin^2\omega = \frac{n^2-b^2}{a^2-b^2}, \cos^2\omega = \frac{a^2-n^2}{a^2-b^2} = \frac{m^2-b^2}{a^2-b^2}. \tag{41}$$

With use of the inverse transformation of (34), we obtain the equation of the ellipse in $x-y$ coordinate:

$$Ax^2 + 2Hxy + By^2 = a^2b^2,$$

where

$$A = b^2\cos^2\omega + a^2\sin^2\omega, B = b^2\sin^2\omega + a^2\cos^2\omega, H = (b^2-a^2)\sin\omega\cos\omega. \tag{42}$$

By substituting (41) into (42), we get

$$A = n^2, B = m^2, H = \pm\sqrt{(n^2-b^2)(m^2-b^2)} = \pm\sqrt{m^2n^2-a^2b^2},$$

where minus sign for $0 \le \omega < \pi/2$ and plus sign for $-\pi/2 \le \omega < 0$.

□

Ajima's theorem

Theorem 1. *An ellipse with axes a and b is inscribed in a rectangle with sides of $2m$ and $2n$. The radius of an circle, which is contact with both the ellipse and the rectangle is given by*

$$r = s - \sqrt{s^2 - h},$$

$$s = m + n + \sqrt{k},$$

$$h = mn(1+\epsilon), k = mn(1-\epsilon),$$

$$\epsilon = \sqrt{1-(ab/mn)^2}.$$

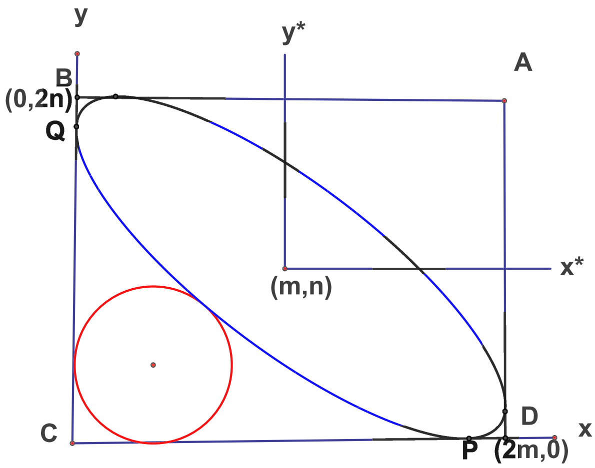

Figure 5: Ajima's Theorem, P(α, 0), Q(0, β)

Proof. We choose CD as x axis and CB as y axis and set the contact point of the ellipse with x and y axes as (α,0) and (0,β). The ellipse is given by

$$k^2\left(\frac{x}{\alpha}+\frac{y}{\beta}-1\right)^2-4xy=0. \tag{43}$$

The circle, which is tangent to x and y axes, is

$$k'^2\left(\frac{x}{r}+\frac{y}{r}-1\right)^2-4xy,\ \ k'=\sqrt{2}r. \tag{44}$$

From Lemma 2 the condition that the circle (44) is tangent to the ellipse (43)

$$2k^2r^2\left(\frac{1}{r}-\frac{1}{\alpha}\right)\left(\frac{1}{r}-\frac{1}{\beta}\right)=(k+\sqrt{2}r)^2. \tag{45}$$

Since the ellipse is tangent to the line of $x=2m$, the following equation has equal roots:

$$\frac{k^2}{\beta^2}y^2+\frac{2k^2}{\beta}\left(\frac{2m}{\alpha}-8m\right)y+k^2\left(\frac{m}{\alpha}-1\right)^2=0. \tag{46}$$

From the condition that the discriminant of (46) is zero, we get

$$\beta=k^2\left(-\frac{1}{2m}+\frac{1}{\alpha}\right). \tag{47}$$

From the condition that the ellipse is tangent to the line of $y=2n$, we have

$$\alpha=k^2\left(-\frac{1}{2n}+\frac{1}{\beta}\right). \tag{48}$$

From (47) and (48), we have $\alpha/m=\beta/n$ and define the parameter t instead of k

$$\alpha=mt,\beta=nt, \tag{49}$$

and then get the relation between k and t from (47):

$$k^2=\frac{2mnt^2}{2-t} \tag{50}$$

Substituting (49) and (50) into (45) we obtain

$$r^2-2(m+n+\sqrt{mn(2-t)})r+mnt=0. \tag{51}$$

By changing the origin to the center of the rectangle we get the equation of the ellipse in $X - Y$ coordinate (see Figure 5):

$$n^2X^2 + m^2Y^2 + 2mn(t-1)XY = m^2n^2t(2-t). \tag{52}$$

With use of Lemma 3, we have the equation of the ellipse in terms of a, b, m, and n:

$$n^2X^2 + m^2Y^2 + 2\sqrt{m^2n^2 - a^2b^2}XY = a^2b^2. \tag{53}$$

Since the equation (52) is equivalent with (53), we have

$$t = 1 + \epsilon, \epsilon = \sqrt{1 - \left(\frac{ab}{mn}\right)^2}, \tag{54}$$

$$t(2-t) = \left(\frac{ab}{mn}\right)^2. \tag{55}$$

The equation (54) satisfies the condition (55). Substituting (54) into (51), we get

$$r = s \pm \sqrt{s^2 - mn(1+\epsilon)}, s = m + n + \sqrt{mn(1-\epsilon)}. \tag{56}$$

Since the circle is located inside the triangle, we choose the minus case of the equation (56).

□

Ajima Naonobu(1732-1798) presented the above theorem as No. 11 problem in [1] and Murata Tunemitu(?-1870) showed the proof given by Ajima in the appendix of [4]. This theorem was adopted as the formula no. 98 in [5], which collected formulae in the field of geometry but did not give their proofs. Formulas Traditional Japanese Geometry (2010) edited by Nakamura Nobuya gives the proofs of all formulas in [5], which is available from the Part 5 of the following home page: `http://www.wasan.jp/kosiki/kosiki.html`

This paper is a revised version of [3] Kinoshita 2012.

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