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Solution to Problem 2018-3-2

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Abstract. A generalization of Problem 2018-3-2 in [5] is given.

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1. Introduction

We give a solution to Problems 2018-3 Problem 2 by giving a generalization of the problem [5]. The problem is as follows (see Figure 1):

Problem 1. For two intersecting circles δ_1 and δ_2 of radius 6, there are four congruent smaller circles such that two of them touch each other and δ_1 and δ_2 internally, each of the other two circles touches δ_1 and δ_2 externally and one of the external common tangents of δ_1 and δ_2. Find the radius of the smaller circles.

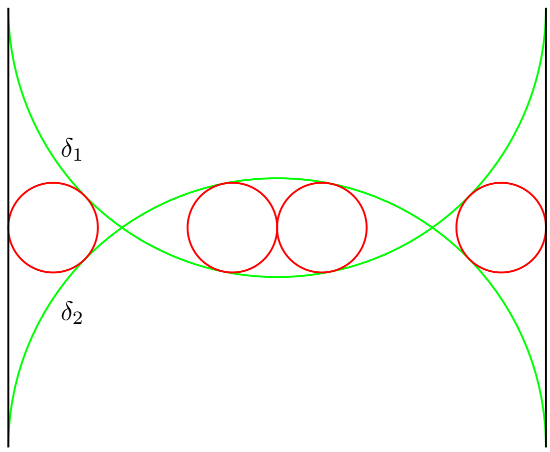

Figure 1.

The same problem can also be found in [6, 7, 8, 9], which are not referred in [5].

2. Generalization

We generalize the problem. Let γ_1, γ_2, $\cdots$, γ_n be congruent circles of radius r touching a line t from the same side such that γ_1 and γ_2 touch and γ_i touches γ_{i-1} at the farthest point on γ_{i-1} from γ_1 for $i = 3, 4, \cdots, n$. In this case we call γ_1, γ_2, $\cdots$, γ_n *congruent circles on a line or congruent circles of radius* r *on* t (see Figure 2).

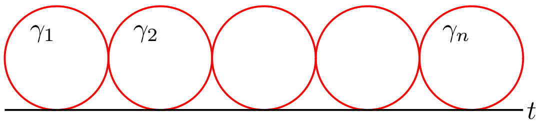

Figure 2.

The problem is generalized as follows (see Figure 3).

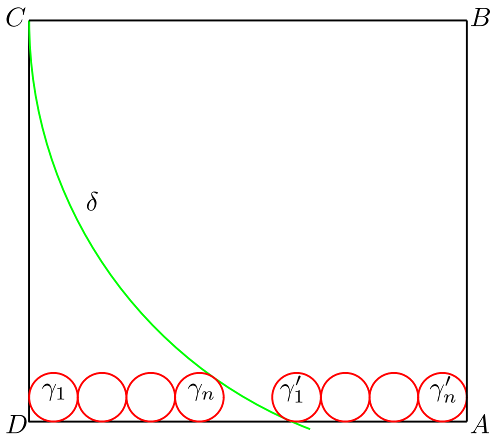

Figure 3: $n = 4$

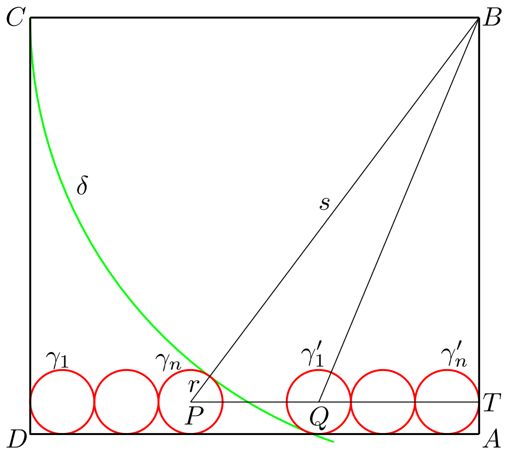

Figure 4: $n = 3$

Theorem 1. *For a rectangle* $ABCD$ *satisfying* $s = |BC| > |AB|$*, let* δ *be the circle with center* B *passing through* C*. If* γ_1, γ_2, $\cdots$, γ_n *are congruent circles of radius* r *on* DA *such that* γ_1 *touches the side* CD *from the same side as* A *and* γ_n *touches* δ *externally from the side opposite to* A*, and* γ'_1, γ'_2, $\cdots$, γ'_n *are congruent circles of radius* r *on* DA *such that* γ'_1 *touches* δ *internally from the side opposite to* D *and* γ'_n *touches the side* AB *from the same side as* D*, then the following statements hold.*
(i) $s = 2(2n + 1)r$.
(ii) *There is a circle of radius* r *touching* DA *and* γ_n *and* γ'_1 *externally.*

Proof. We assume that P and Q are the centers of γ_n and γ'_1, respectively, and T is the point of tangency of γ'_n and AB (see Figure 4). From the right triangles BPT and BQT, we get

$$(s + r)^2 - (s - (2n - 1)r)^2 = (s - r)^2 - ((2n - 1)r)^2. \tag{1}$$

Solving the equation for s, we get (i). The part (ii) follows from (i). □

Drawing Figure 4 with its images by the reflections in the lines AB and PQ and removing several line segments from the resulting figure, we get Figure 5. Therefore Theorem 1 is a generalization of Problem 1, which is the case $n = 1$. If γ is the circle in (ii), the fact shows that the $2n + 1$ circles γ_1, γ_2, $\cdots$, γ_n, γ, γ'_1,

$\gamma'_2, \cdots, \gamma'_n$ form congruent circles of radius r on DA. Since $|BT|^2$ equals the both sides of (1), we have $|BT| = 2\sqrt{3n(n+1)}r$.

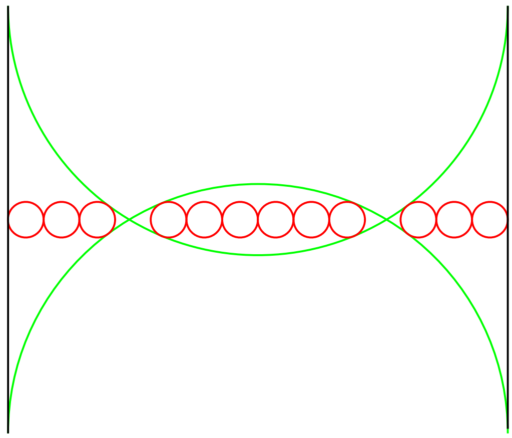

Figure 5.

Let us assume that a positive integer n and a positive real number r are given. For a rectangle $ABCD$ satisfying $|AB| = (2\sqrt{3n(n+1)}+1)r$ and $|DA| = 2(2n+1)r$, let $\gamma_1, \gamma_2, \cdots, \gamma_{2n+1}$ be congruent circles of radius r on DA, such that γ_1 touches the side CD from the same side as A and γ_{2n+1} touches the side AB from the same side as D. Then the circle with center B passing through C touches the circles γ_n externally and γ_{n+2} internally by the uniqueness of the figure.

For more properties on congruent circles on a line, see [1, 2, 3, 4].

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