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Inscribed rectangles in a kite and a solution to 2017-1 Problem 5

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Abstract. Several properties of inscribed rectangles in a kite are studied, which give a solution to 2017-1 Problem 5.

Keywords. sangaku, rectangle inscribed into a kite, square inscribed into a kite, regular convex polygon.

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1. Introduction

In this article we give several properties of an inscribed rectangle in a kite, which we believe to be new. Then we give a a solution of Problem 5 in [1]. The problem is given only by a figure with no text (see Figure 1), which may be stated as follows:

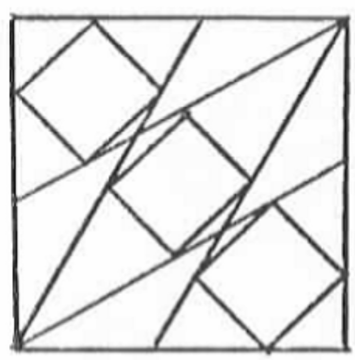

Figure 1. Problem

Problem 1. For a square $ABCD$ of side length 1 with center S, let M and N be points on the diagonal AC such that S is the midpoint of MN. The inscribed square of the convex quadrangle formed by the lines AB, AD, BM and DM is congruent to the inscribed square of the rhombus $BNDM$.
(i) Find $|MS|$.
(ii) Find the side of the smaller squares.

A problem with the same figure can be found in [2]. It states that the relation $l = \frac{5+\sqrt{17}}{2\sqrt{2}}s$ holds, where l and s are the side lengths of the large square and the small squares, respectively.

2. Solution

We solve the problem with the assumption that the small squares are symmetric in the line AC as shown in Figure 2. We will prove our assumption is true in Section 4.

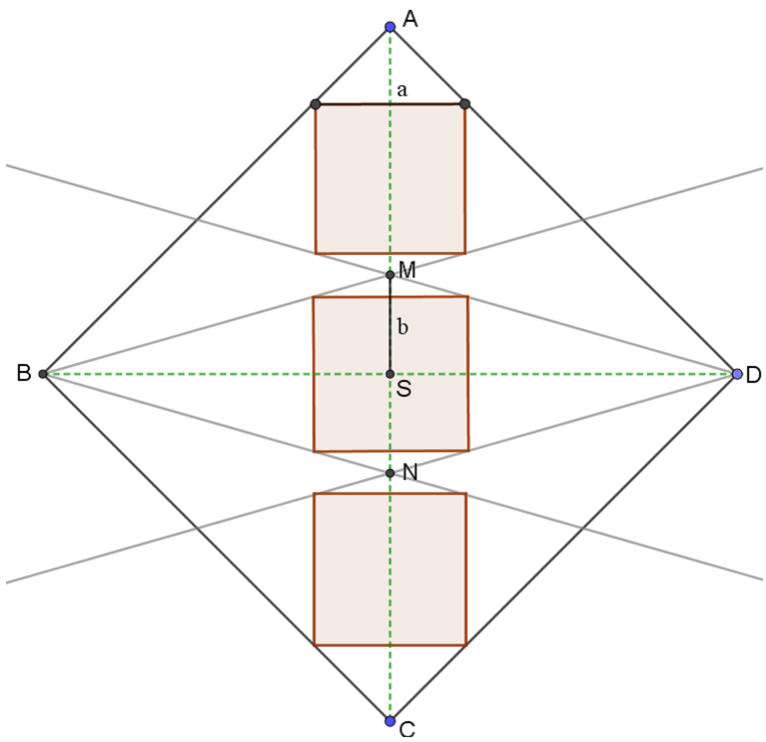

Figure 2. The solution

Let a be the side of the smaller squares and $b = |MS|$. Then the following two equations hold:

$$\frac{\sqrt{2}}{2} = \frac{a}{2} + \frac{a}{2} + 2b,$$
$$\frac{b}{\frac{\sqrt{2}}{2}} = \frac{b - \frac{a}{2}}{\frac{a}{2}}.$$

The first equation is derived from length of the segment AS. The second is derived from the similarity of the triangle MSD and the tiny triangle with corner M and the side of the inscribed square of $BMDN$ lying the side of BD opposite to N.

Solution is: $a = \frac{5-\sqrt{17}}{2\sqrt{2}}$ and $b = \frac{\sqrt{17}-3}{4\sqrt{2}}$. So the points M, N could be found only by a ruler and compasses. The relation $a = \frac{5-\sqrt{17}}{2\sqrt{2}}$ is essentially the same to that given in [2].

3. Generalization

Let $EFGH$ be a kite with equal-length sides EF and HE. If $P_1P_2\cdots P_{2n+1}$ is a regular $2n+1$-gon such that the vertices P_{n+1} and P_{n+2} lie on HE and EF, respectively, and $P_1 = G$, we say $P_1P_2\cdots P_{2n+1}$ is semi-inscribed in $EFGH$. If $P_1P_2\cdots P_{2n}$ is a regular $2n$-gon such that the vertices P_n and P_{n+1} lie on HE and EF, respectively, and the vertices P_{2n} and P_1 lie on FG and GH, respectively and $P_1P_{2n} \perp EG$, we say $P_1P_2\cdots P_{2n}$ is semi-inscribed in $EFGH$ (see Figure 3). Problem 1 is generalized as follows:

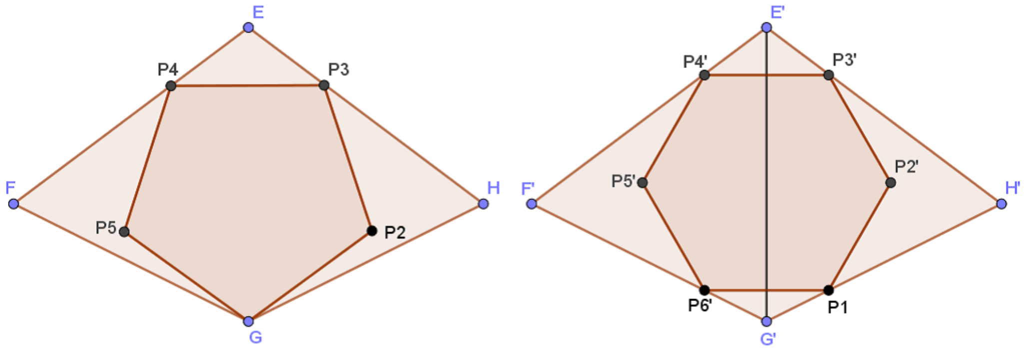

FIGURE 3. Semi-inscribed regular polygon

Theorem 1. *For a square $ABCD$ of side length 1 with center S, let M and N be points on the diagonal AC such that S is the midpoint of MN. The semi-inscribed regular n-gon in the convex quadrangle formed by the lines AB, AD, BM and DM so that M coincides with one of the vertices of the n-gon if n is odd, is congruent to the semi-inscribed regular n-gon in the rhombus $BNDM$.*
If n is odd, and $u_n = \cos\frac{\pi}{2n}$, the following statements hold.

(i) *The side length of the regular n-gon equals* $\dfrac{3}{\sqrt{2}} - \sqrt{\dfrac{9u_n+1}{2(u_n+1)}}$.

(ii) $|MS| = \dfrac{\sqrt{(u_n+1)(9u_n+1)} - 1 - 3u_n}{2\sqrt{2}}$,

If n is even and $u_n = \cot(\pi/n)$, the following statements hold.

(iii) *The side length of the regular n-gon equals* $\dfrac{3}{\sqrt{2}} - \dfrac{1+\sqrt{(9u_n+8)u_n}}{\sqrt{2}(u_n+1)}$,

(iv) $|MS| = \dfrac{\sqrt{(9u_n+8)u_n} - 3u_n}{4\sqrt{2}}$.

Proof. Let a be the side length of the regular n-gon and $b = |MS|$. If n is odd and $u_n = \cos\frac{\pi}{2n}$, we have

$$\frac{\sqrt{2}}{2} = \frac{a}{2} + \frac{au_n}{2} + b,$$
$$\frac{b}{\frac{\sqrt{2}}{2}} = \frac{2b - \frac{au_n}{2}}{\frac{a}{2}}.$$

Solving the equations, we get (i) and (ii). If n is even and $u_n = \cot(\pi/n)$, we have

$$\frac{\sqrt{2}}{2} = \frac{a}{2} + \frac{au_n}{2} + 2b,$$
$$\frac{b}{\frac{\sqrt{2}}{2}} = \frac{b - \frac{au_n}{2}}{\frac{a}{2}}.$$

Solving the equations we have (iii) and (iv). □

You can see an example of semi-inscribed regular pentagons in Figure 4.

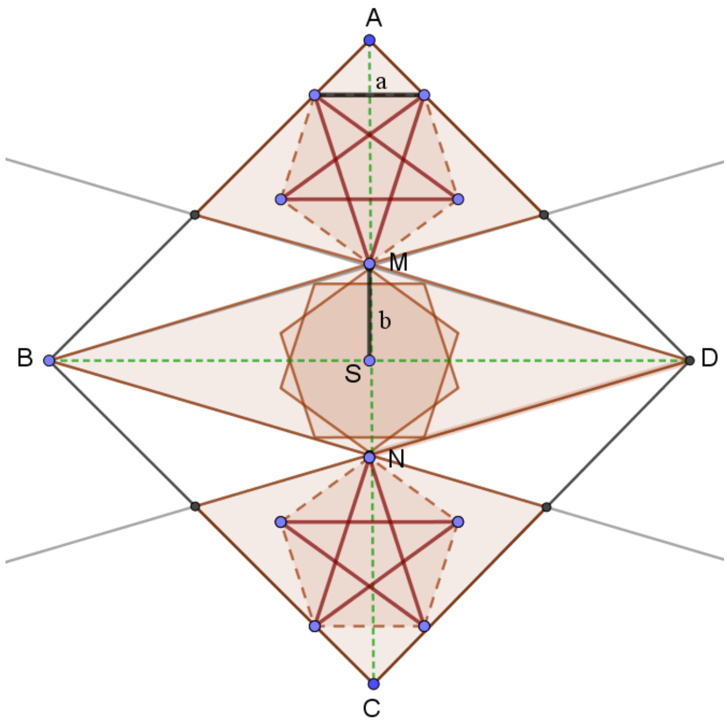

FIGURE 4. Regular pentagons

4. A RECTANGLE INSCRIBED INTO A GIVEN KITE

In this section we will prove that the small squares inscribed into the first and third areas, which are kites given in the original problem must be symmetrical about the diagonal of the kites. To prove this, we prove more generally results given in the following three theorems.

A kite is given in a Cartesian coordinate system in such a way that the point of intersection of its diagonals is the origin and the corners of the kite are: $A(0, a)$, $B(-b, 0)$, $C(0, -c)$, $D(b, 0)$, where $a, b, c > 0$. In this case we call $ABCD$ an a-b-c kite. Then there are two options how to inscribe a rectangle into this kite. First, obvious way is to inscribe it symmetrically about the y-axis - then we have infinitely many solutions including one square. We say that the rectangle is inscribed non-symmetrically in the remaining case. The case is resolved in the following theorems.

Theorem 2. *Let $KLMN$ be a rectangle inscribed into an a-b-c kite $ABCD$ non-symmetrically, where K lies on AB, N lies on DA. Then the difference between the x-coordinates of the points K and N is constant and is equal to*

$$d = \frac{2abc}{b^2 + ac}.$$

Proof. Let us name the x-coordinates of K to be $-u$ and of N to be $u+k$, where $0 < u < b$ and k is the variable. (See Figure 5.) Let m and $-l$ be the x-coordinates of M and L, respectively. Then the y-coordinates of K, L, M and N are $a - ua/b$, $-c + lc/b$, $-c + mc/b$ and $a - (u+k)a/b$, respectively. Therefore we get

$$\overrightarrow{KN} = (x_1, y_1) = \left(2u + k, -\frac{a}{b}k\right),$$
$$\overrightarrow{MN} = (x_2, y_2) = \left(u + k - m, -\frac{a}{b}(u+k) + a + c - \frac{c}{b}m\right),$$
$$\overrightarrow{LK} = (x_3, y_3) = \left(-u + l, -\frac{a}{b}u + a + c - \frac{c}{b}l\right).$$

Eliminating l from $x_1x_2 + y_1y_2 = 0$, $x_2 = x_3$, $y_2 = y_3$ and solving the resulting equations for k and m, we get $(k, m) = (0, u)$ or

$$k = -2u + d, \tag{1}$$

$$m = \frac{ab(c-a)}{b^2 + ac} + \frac{au}{c}. \tag{2}$$

□

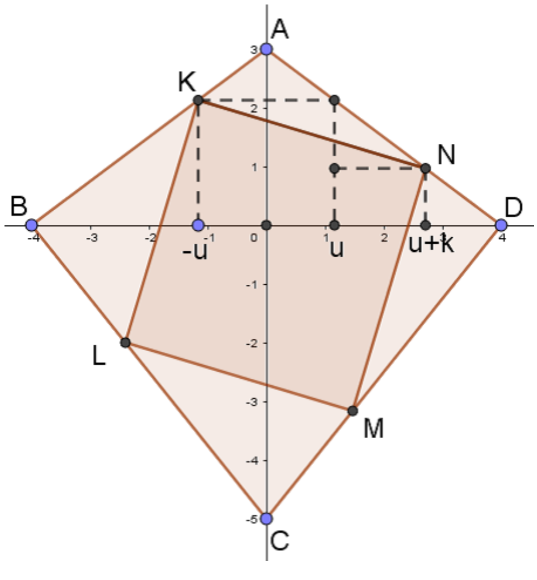

FIGURE 5. A rectangle inscribed into a kite

Theorem 3. *If a rectangle is inscribed in an a-b-c kite non-symmetrically, then the ratio of lengths of adjacent sides of the rectangle is constant and equals*

$$q = \frac{(a+c)b}{2ac}.$$

Proof. We assume that a rectangle $KLMN$ is inscribed in an a-b-c kite $ABCD$ as in Theorem 2 and use the same notations as in the proof of Theorem 2. Using (1) and (2), we get

$$|KN|^2 = x_1^2 + y_1^2 = \frac{4a^2 N_P}{b^2(b^2+ac)^2},$$

$$|MN|^2 = x_2^2 + y_2^2 = \frac{(a+c)^2 N_P}{c^2(b^2+ac)^2},$$

where $N_P = a^2c^2(b-u)^2 + 2ab^2cu(-b+u) + b^4(c^2+u^2)$. This proves the theorem. $\square$

Notice that Theorem 3 shows that inscribed rectangles in a given kite non-symmetrically are all similar. In the following theorem we shall prove that there are infinitely many similar rectangles inscribed in a given kite. We will use only the case where $a < c$ because for $a > c$ the similar theorem holds. (We could rename the corners.) Variable d is the distance of x-coordinate of points K and N from Theorem 2.

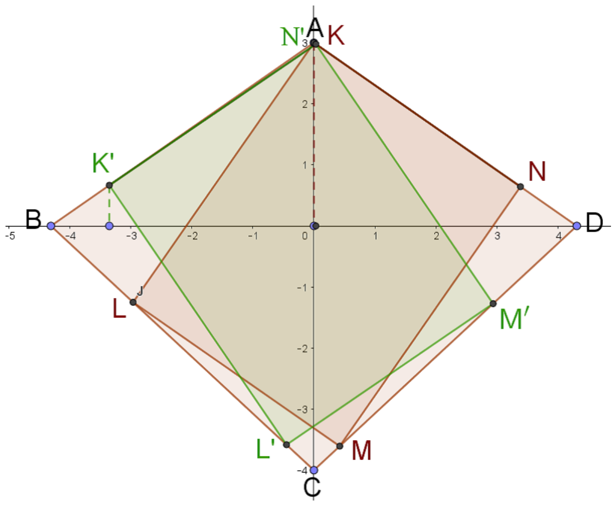

FIGURE 6. Bounds for $b^2 \geq ac$

Theorem 4. *If an $a-b-c$ kite $ABCD$ is given with $a < c$, then it is possible to inscribe a rectangle $KLMN$ non-symmetrically, if for x-coordinate of the point K lying on the side AB holds $k = -u$, where*

if $b^2 \geq ac$, then $0 < u < d$,

if $b^2 < ac$, then $d - b < u < b$.

Proof. From Theorem 2 we get that the difference between the x-coordinates of K and N equals to $d = \frac{2abc}{b^2+ac}$. Now we can see, that there are two options for d. First is $d \leq b$ which is the same as $b^2 \geq ac$ (see Figure 6). Second is $d > b$ which is the same as $b^2 < ac$ (see Figure 7). Only from this fact we have the bounds for x-coordinates of K or N. What remains is to prove that also x-coordinates of M and L lie in the same intervals. In the proof of Theorem 2 we get the x-coordinate of the point M: $m = \frac{ab(c-a)}{b^2+ac} + \frac{a}{c} \cdot u$.

For the case $b^2 \geq ac$ we have $0 < u < d$ which should improve

$$0 < m < d \Leftrightarrow \frac{-bc(c-a)}{b^2+ac} < u < \frac{bc(c+a)}{b^2+ac}$$

Since $\frac{-bc(c-a)}{b^2+ac} < 0$ and also $\frac{bc(c+a)}{b^2+ac} > d$ for $c > a$, we have the desired implication.
For $b^2 < ac$ it holds $d - b < u < b$ and we should prove that also

$$d - b < m < b \Leftrightarrow \frac{bc(c+a)}{b^2+ac} - \frac{bc}{a} < u < \frac{bc}{a} - \frac{bc(c-a)}{b^2+ac}$$

Since $\frac{bc(c+a)}{b^2+ac} - \frac{bc}{a} < d - b$ and also $\frac{bc}{a} - \frac{bc(c-a)}{b^2+ac} > b$ for $c > a$, we have the desired implication. □

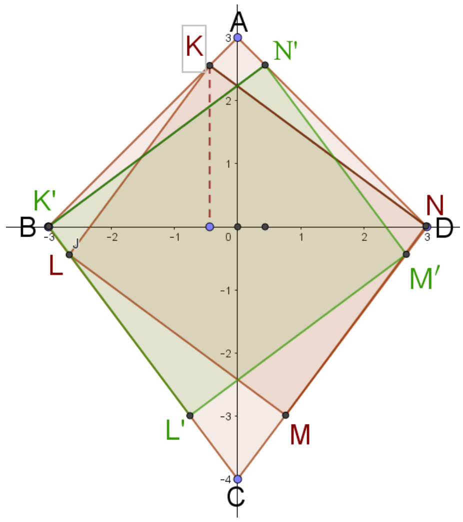

FIGURE 7. Bound for $b^2 < ac$

Notice that length of both intervals are nonzero, so there are infinitely many inscribed rectangles in an arbitrary kite.

Corollary 1. *If and only if $2ac = b(a + c)$ for an a-b-c kite $ABCD$, there is possible to inscribe infinitely many different squares in $ABCD$.*

In the original problem, the first and third areas are kites with $a = b$. From the corollary it follows that c must be equal to a but it is not, therefore there are no non-symmetrical inscribed square.

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