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A note on the arbelos in Wasan geometry, Satoh's problem

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Abstract. We generalize a sangaku problem involving an arbelos proposed by Satoh, and show the existence of three congruent non-Archimedean circles.

Keywords. arbelos, non-Archimedean congruent circles.

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1. Introduction

We consider the arbelos appeared in Wasan geometry, and consider an arbelos formed by three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB (see Figure 1). For a point F on the segment AO, let h be the perpendicular to AB at F. The circle with diameter FO is denoted by δ_1, and the incircle of the curvilinear triangle made by α, γ and h is denoted by δ_2. Let r_i be the radius of δ_i. The incircle of the curvilinear triangle made by α, γ and the external common tangent of α and β is denoted by ε. Let e be the radius of ε. In this note we generalize the following problem in the sangaku hung in Fukushima in 1883 proposed by Satoh (佐藤金之明) [2] (see Figure 2).

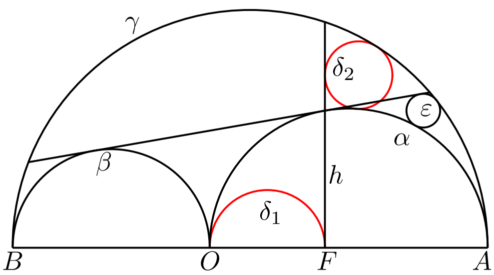

Figure 1.

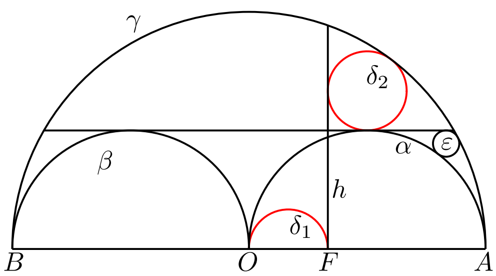

Figure 2.

Problem 1. If α and β are congruent and δ_1 and δ_2 are also congruent, show $r_1 = 3e$.

2. Generalization

We give a condition that the circles δ_1 and δ_2 are congruent, and show the existence of two pairs of three congruent circles. The radii of α and β are denoted by a and b, respectively. Let the tangent of α from B touch α at a point T, and let U be the foot of perpendicular from T to AB and $u = ab/(a + 2b)$ (see Figure 3).

Proposition 1. $|OU| = 2u$.

Proof. If C is the center of α, the triangles BCT and TCU are similar. Hence we get $(a - |OU|)/a = a/(a + 2b)$, which implies $|OU| = 2u$. □

Theorem 1. *The circles δ_1 and δ_2 are congruent if and only if h passes through T. In this event the common radius equals u.*

Proof. Let $k = |FO|$. Since r_2 is proportional to the distance between the center of δ_2 and the radical axis of α and γ [1, p. 108], $r_2/(2a - k) = b/(2a + 2b)$, i.e., $r_2 = (2a-k)b/(2a+2b)$, while $r_1 = k/2$. Therefore $r_1 = r_2$ is equivalent to $k = 2u$ (see Figure 4). □

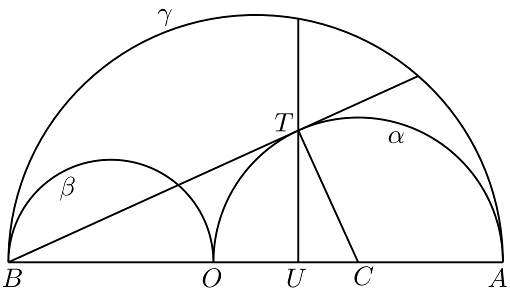

Figure 3.

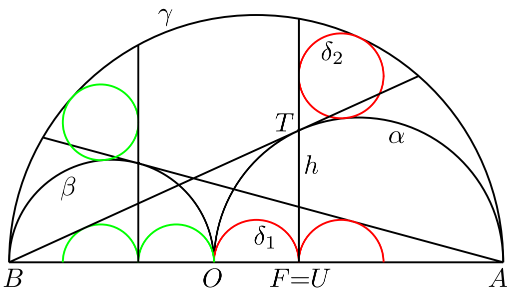

Figure 4.

Since $e = a^2b/(a + 2b)^2$ [3, 4], we get $u = e(a + 2b)/a$. This gives a solution of Problem 1 in the case $a \neq b$ and implies $u = 3e$ in the case $a = b$. The circle touching the tangent of β from A and h at F from the side opposite to B is congruent to δ_2 [3]. Circles of radius $ab/(a + b)$ are said to be Archimedean. Therefore we get three congruent non-Archimedean circles of radius u in the case $F = U$. Exchanging the roles of α and β we get another three congruent circles of radius $ab/(2a + b)$, which are denoted by the green circles in Figure 4.

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