Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 4 (2020), pp. 1-8
Received 10 January 2020. Published on-line 14 January 2020
web: http://www.sangaku-journal.eu/

A four circle problem and division by zero

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We generalize a problem involving four circles and a triangle, and consider some limiting cases of the problem by division by zero.

Keywords. four circle problem, division by zero.

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

We generalize the following problem involving four circles and a triangle in [20] (see Figure 1). The same sangaku problem was proposed in 1826 and cited in [19] and [1] with no solution. Some limiting cases of the problem will be considered by division by zero [6].

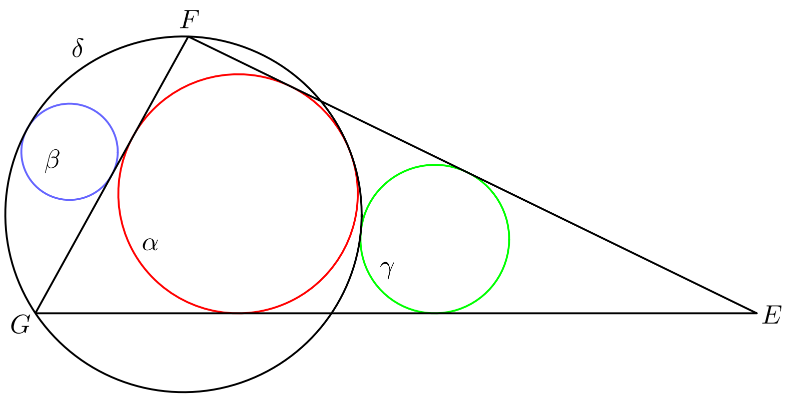

Figure 1.

Problem 1. For a triangle EFG with incircle α, δ is the circle passing through E and F and touching α, γ is the incircle of the curvilinear triangle made by δ and the sides EF and GE, and β is the circle touching δ and FG at the midpoint from the side opposite to α. Let a, b and c be the radii of α, β and γ, respectively. Show $a^2 = 4bc$.

A similar sangaku problem considering the case $|EF| = |GE|$ can be found in [2, p. 302].

2. Generalization

The problem assumes that α is the incircle of EFG, but we show that the condition is inessential. We consider the following figure (see Figure 2): For a chord FG of a circle δ, M is the midpoint of FG, β is a circle touching δ and FG at M, α is a circle touching δ and the chord FG from the side opposite to β, f and g are the tangents of α from the points F and G, respectively, γ is the circle lying on the same side of FG as α and touching δ externally and f and g from the same side as α. Let a, b, c and d be the radii of α, β, γ and δ, respectively. We denote this configuration by $\mathcal{S}$.

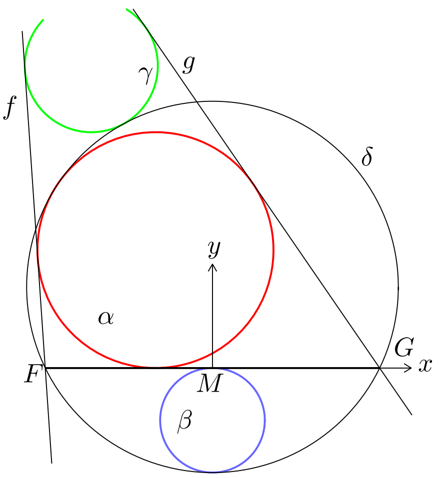

Figure 2: The configuration $\mathcal{S}$.

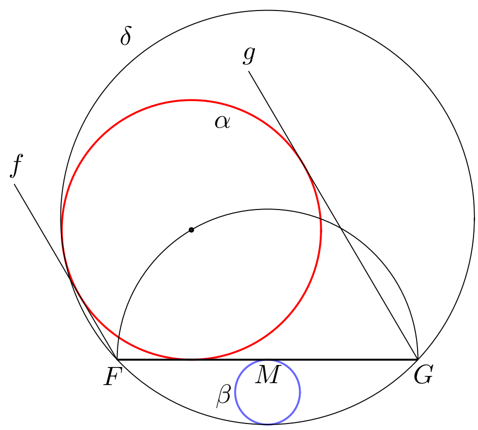

Figure 3: $4b = a = c$

We use a rectangular coordinate system with origin M such that the center of α has coordinate (x_a, a) for a real number x_a. Firstly we consider the special case in which f and g are parallel (see Figure 3).

Theorem 1. *The following statements are equivalent for* $\mathcal{S}$.
(i) *The lines f and g are parallel.*
(ii) *The center of α lies on the circle of diameter FG.*
(iii) $a = 4b$.

Proof. We may assume that the point G has coordinates $(k, 0)$, and f and g have equations $x + m_1 y + k = 0$ and $x + m_2 y - k = 0$, respectively for real numbers m_1 and m_2. Since f and g touch α, we have

$$m_1 = \frac{a^2 - (k + x_a)^2}{2a(k + x_a)}, \quad m_2 = -\frac{a^2 - (k - x_a)^2}{2a(k - x_a)}. \tag{1}$$

Notice that $k^2 - x_a^2 \neq 0$, since $k^2 - x_a^2 = 0$ implies that α touches FG at F or G. The lines f and g are parallel if and only if $m_1 = m_2$, which is equivalent to

$$a^2 + x_a^2 = k^2. \tag{2}$$

This proves the equivalence of (i) and (ii), since the left side equals the square of the distance between the center of α and M (see Figure 3). While the square of the distance between the centers of δ and α equals

$$x_a^2 + (d - 2b - a)^2 = (d - a)^2. \tag{3}$$

While the intersecting chords theorem gives

$$2b(2d - 2b) = k^2. \tag{4}$$

Eliminating d from (3) and (4), we get $xa^2 + 4ab = k^2$, which implies

$$a^2 + xa^2 - k^2 = a(a - 4b).$$

Hence (2) and $a = 4b$ are equivalent, i.e., (i) and (iii) are equivalent. □

Transforming the configuration $\mathcal{S}$ continually, we get the next corollary.

Corollary 1. *The relation $4b < a < c$ or $4b = a = c$ or $4b > a > c$ holds for $\mathcal{S}$.*

Figures 2, 3 and 4 show the cases $4b > a > c$, $4b = a = c$ and $4b < a < c$, respectively. The next theorem is a generalization of Problem 1.

Theorem 2. *The following statements hold.*
(i) *The relation $a^2 = 4bc$ holds.*
(ii) *One of the internal common tangents of α and γ is parallel to FG.*

Proof. We use the same notation as in the proof of Theorem 1. If f and g are parallel, we get $a = c$. Therefore we get $a^2 = 4bc$ by Theorem 1. We assume that f and g intersect. We denote the point of intersection by E, which has coordinates

$$(x_e, y_e) = \left(\frac{k(m_1 + m_2)}{m_1 - m_2}, \frac{-2k}{m_1 - m_2}\right). \tag{5}$$

Substituting (1) in (5), we get

$$(x_e, y_e) = \left(x_a - \frac{2a^2 x_a}{a^2 - k^2 + x_a^2}, 2a - \frac{2a^3}{a^2 - k^2 + x_a^2}\right). \tag{6}$$

The square of the distance between the centers of δ and γ equals

$$x_c^2 + (d - 2b - y_c)^2 = (c + d)^2, \tag{7}$$

where (x_c, y_c) are the coordinates of the center of γ. Since E is the external center of similitude of α and γ, we get

$$\frac{-cx_a + ax_c}{a - c} = x_e, \quad \frac{-ca + ay_c}{a - c} = y_e. \tag{8}$$

Eliminating x_a, x_c, y_c, x_e, y_e and d from (3), (4), (6), (7), (8), we get

$$(a^2 - 4bc)j(k) = 0,$$

where $j(k) = 4(a - 4b)b^2 - (4b - c)k^2$. If $j(k) = 0$, we have $k^2 = 4(a - 4b)b^2/(4b - c) > 0$. This implies $a < 4b < c$ or $c < 4b < a$. However this contradicts Corollary 1. Therefore we get $j(k) \neq 0$, which implies $a^2 = 4bc$.

We prove (ii). If f and g are parallel, the centers of α, γ and M are collinear, i.e., $x_a/a = x_c/y_c$. Eliminating b, c, k, x_a, x_c from the equations $x_a/a = x_c/y_c$, $a = c$, $a = 4b$, (3), (4) and (7), we get

$$(3a - y_c)((a + 4d)a + (4d - a)y_c) = 0.$$

Therefore we get $y_c = 3a = 2a + c$, since $(4d - a)y_c > 0$. If f and g intersect, we eliminate b, k, x_a, x_c, x_e, y_e from (3), (4), (6), (7), (8). Then we get

$$(2a + c - y_c)((a + 4d)c + (4d - a)y_c) = 0.$$

Therefore we get $y_c = 2a + c$. This proves (ii). □

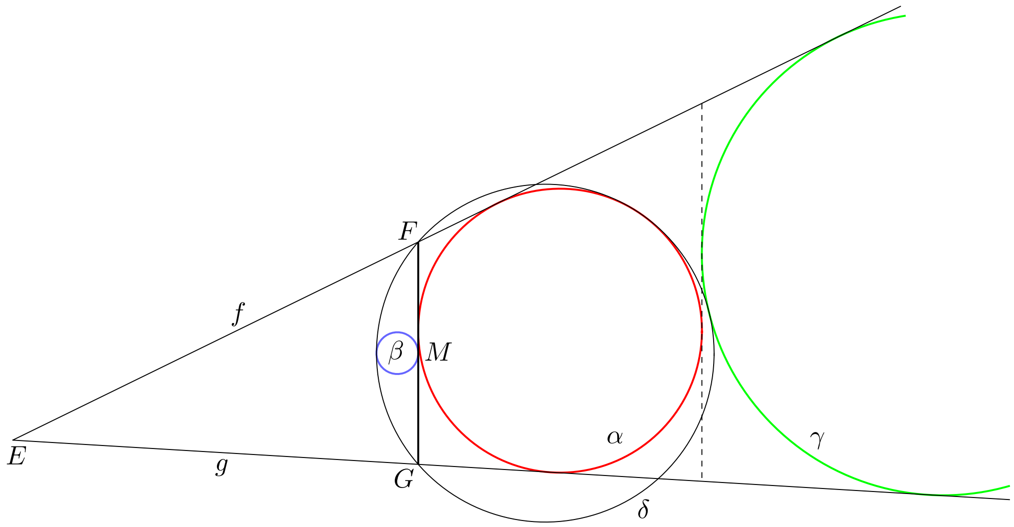

Figure 4: The configuration $\mathcal{S}$ in the case $4b < a < c$.

There are several sangaku problems stating the next corollary [2, p. 312, p. 317, p. 419] (see Figure 5).

Corollary 2. *For a semicircle δ with diameter FG, let α be the circle of radius a touching δ and FG at the midpoint. If c is the inradius of the curvilinear triangle made by δ and the tangents of α from the points E and F, then $a = 4c$.*

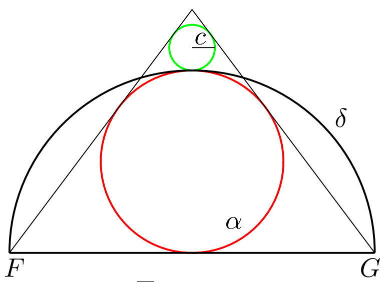

Figure 5.

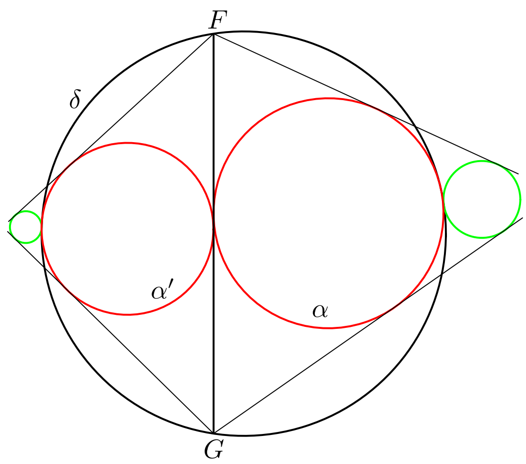

Figure 6.

The next corollary can be found in the sangaku hung in 1830 [3, p. 40], which is incorrectly cited in [1, p. 34] (see Figure 6).

Corollary 3. *For the configuration $\mathcal{S}$, let α' be a circle of radius a' touching the circle δ and its chord FG from the side opposite to α. If the circle lying on the same side of FG as α' and touching δ externally and the tangents of α' from F and G from the same side as α' has radius c', then $a^2a'^2 = cc'|FG|^2$.*

Proof. Let b' be the radius of the circle touching δ and FG at the midpoint from the side opposite to α'. Then we have $a'^2 = 4b'c'$, while $|FG|^2 = 16bb'$ and $a^2 = 4bc$. Eliminating b and b' from the three equations, we get $a^2a'^2 = |FG|^2cc'$. □

3. Limitimg cases with division by zero

In this section we fix the circle δ for $\mathcal{S}$ and consider the case where one of the circles α and β has radius 0 with the definition of division by zero [6]:

$$\frac{z}{0} = 0 \ \textit{for a complex number } z. \tag{9}$$

Notice that the definition implies that lines have radius 0 as circles [17].

We now consider the case in which the figure is symmetric in the perpendicular bisector of FG and use a rectangular coordinate system with origin at the point of tangency of β and δ such that the center of δ has coordinates $(0, d)$. The point of tangency of γ and δ and the tangent of δ at the point are denoted by D and t, respectively (see Figure 7). Notice that $d = a + b$.

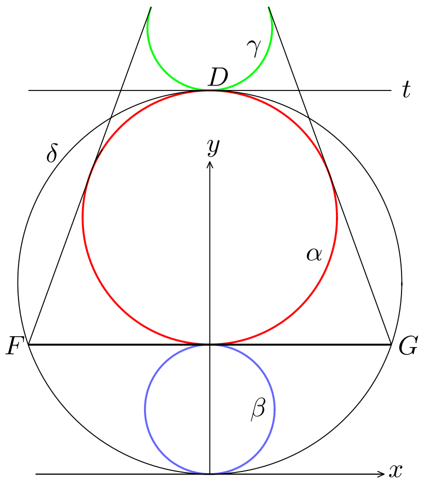

Figure 7.

3.1. **The case $b = 0$.** Firstly we consider the case $b = 0$. Then β is a point or a line. The circle α has an equation $x^2 + (y - (b + d))^2 = (b - d)^2$, which is arranged as

$$f_a(x, y) = (x^2 + (y - d)^2 - d^2) + 2b(2d - y) = 0. \tag{10}$$

From $f_a = 0$, we get $x^2 + (y - d)^2 = d^2$ in the case $b = 0$. Also from $f_a/b = 0$ we get $y = 2d$ in the case $b = 0$ by (9). Hence α *coincides with the circle* δ *or the line* t in the case $b = 0$.

The circle β has an equation

$$f_b(x, y) = (x^2 + y^2) - 2by = 0.$$

From $f_b = 0$ we get $x^2 + y^2 = 0$ in the case $b = 0$. Also from $f_b/b = 0$ we get $y = 0$ in the case $b = 0$ by (9). Hence β *coincides with the origin or the* x*-axis* in this case.

The circle γ has an equation $x^2 + (y - 2d - c)^2 = c^2$, where $c = (d - b)^2/(4b)$ by Theorem 2. Therefore γ has an equation

$$f_c(x, y) = \frac{d^2}{2b}(2d - y) + \left(x^2 + \left(y - \frac{3d}{2}\right)^2 - \frac{d^2}{4}\right) + \frac{b}{2}(2d - y) = 0.$$

From $f_c = 0$ we get $x^2 + (y - 3d/2)^2 = (d/2)^2$ in the case $b = 0$. Also from each of $f_c b = 0$ and $f_c/b = 0$ we get $y = 2d$ in the case $b = 0$. Hence γ *coincides with the line* t *or the circle of radius* $d/2$ *touching* δ *at* D in this case.

When β approaches to the origin, the circles α and γ approach to δ and t, respectively. Therefore we can consider that α and γ coincide with δ and t, respectively when β degenerates to the origin, (see Figure 8). The relation $a^2 = 4bc$ does not holds in this case, but $a^2/b = 4c$ and $a^2/c = 4b$ hold by (9), since the radius of t equals 0.

When β coincides with the x-axis, we can thereby consider that α and γ coincides with t and the circle of radius $d/2$ touching δ at D, respectively as the remaining case (see Figure 9). The relation $a^2 = 4bc$ holds in this case.

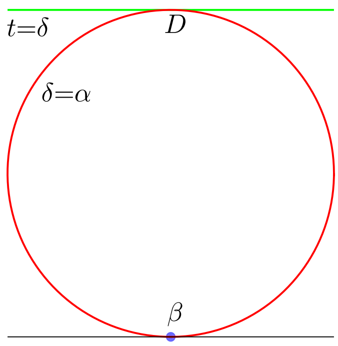

Figure 8.

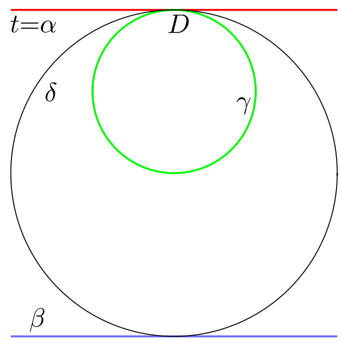

Figure 9.

case	α	β	γ	relation of the radii
1	δ	origin	t	$a^2/b = 4c$, $a^2/c = 4b$
2	t	x-axis	circle of radius $d/2$ touching δ at D	$a^2 = 4bc$

Table 1: $b = 0$.

We summarize the results in Table 1. The case 1 described in Figure 8 is supposable without (9). But (9) enable us to get the case by algebraic manipulation. On the other hand, the case 2 described in Figure 9 can not be obtained without (9). In this case $d = a + b$ does not hold, but still can be considered that α and β touch. However we cannot attain a reasoned interpretation for this case at the current moment. A similar phenomenon, in which a circle of half the radius appears, can be found in [8].

3.2. **The case $a = 0$.** We now consider the case $a = 0$. Substituting $b = d - a$ in (10), we get

$$f_a = (x^2 + (y - 2d)^2) + 2a(y - 2d) = 0.$$

Hence we get $x^2 + (y - 2d)^2 = 0$ or $y = 2d$ in the case $a = 0$. Therefore α *coincides with* D *or* t in this case. Similarly we have

$$f_b = (x^2 + (y - d)^2 - d^2) + 2ay = 0.$$

Therefore we get $x^2 + (y - d)^2 = d^2$ or $y = 0$ in the case $a = 0$. Hence β *coincides with* δ *or the* x*-axis* in the case $a = 0$. Also we have

$$f_c = 2d(x^2 + (y - 2d)^2) - 2a(x^2 + (y - 2d)^2) + a^2(2d - y) = 0.$$

Therefore we get $x^2 + (y - 2d)^2 = 0$ or $y = 2d$ in the case $a = 0$. Hence γ *coincides with* D *or* t in this case.

When α approaches to D, β and γ approach to δ and D, respectively. Hence we consider that β and γ coincide with δ and D, respectively when α coincides with D (see Figure 10). As the remaining case β and γ coincide with the x-axis and t, respectively when α coincides with t (see Figure 11).

We summarize the results in Table 2. The case 3 described in Figure 10 is supposable without (9). On the other hand, the case 4 described in Figure 11 can not be obtained without (9). However we cannot attain a reasoned interpretation for this case at the current moment. The relation $a^2 = 4bc$ holds in both cases.

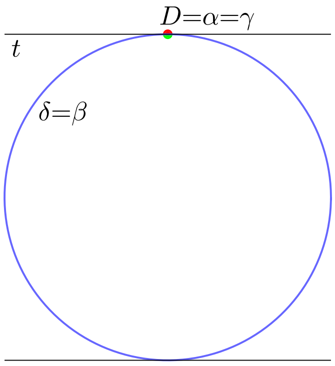

Figure 10.

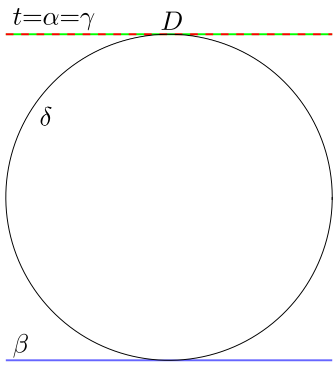

Figure 11.

case	α	β	γ	relation of the radii
3	D	δ	D	$a^2 = 4bc$
4	t	x-axis	t	$a^2 = 4bc$

Table 2: $a = 0$.

For a brief introduction of division by zero with Wasan geometry see [14], and its application to Wasan geometry see [4], [8], [9, 10, 11, 12, 13], [15]. For an extensive reference of division by zero and division by zero calculus, see [17].

4. Incorrect sangaku problems

In [16] we have considered two incorrect sangaku problems in [5, p. 69, p. 123], each of which can also be found in [7] and [21], respectively.

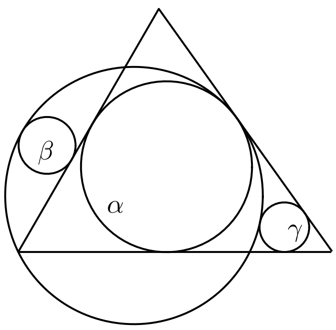

Figure 12: The figures in [5], [21].

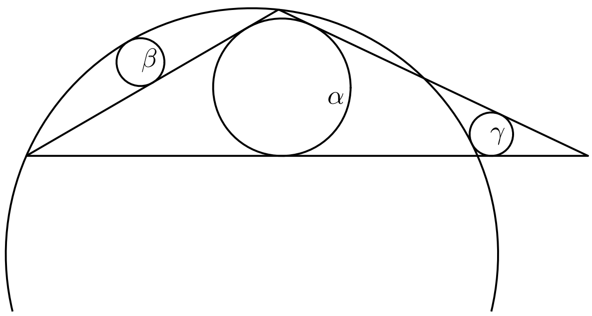

Figure 13: The figure in [5].

The problems and the answers are almost the same as Problem 1, i.e., they demand to show the relation $a^2 = 4bc$ for the three circles α, β and γ of radii a, b and c, respectively. However the figures are slightly different as shown in Figures 12 and 13. And the figure in [21] is also the same as Figure 12. It seems that those problems were correct and essentially the same as Problem 1 in the original context but the figures were incorrectly transcribed in [5] and [21]. While the figure in [7] is the same as Figure 1. Therefore the problem is correct.

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