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Non-Archimedean congruent circles in Tenzan Tebikigusa Furoku

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Abstract. We generalize non-Archimedean congruent circles appeared in Sampō Tenzan Tebikigusa Furoku to the collinear arbelos.

Keywords. collinear arbelos, non-Archimedean congruent pairs of circles.

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1. Introduction

For two points P and Q on a line AB in the plane, we denote the semicircle of diameter PQ by (PQ), where all the semicircles with diameters on AB are constructed on the same side. We consider an arbelos formed by the three semicircles (AO), (BO) and (AB) for a point O on the segment AB, where $|AO| = 2a$ and $|BO| = 2b$ (see Figure 1). The perpendicular to AB at the point O is called the axis. Inradius of the curvilinear triangle made by (AB) and the axis and one of the semicircles (AO) and (BO) equals $ab/(a+b)$, and circles of the same radius are called Archimedean circles of the arbelos.

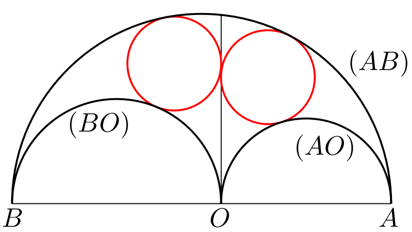

Figure 1.

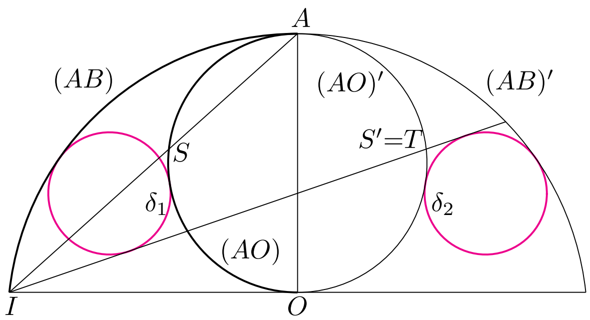

Figure 2.

We denote the reflection in the line AB by $'$. Let I be the point of intersection of (AB) and the axis. In [3], we have shown the following theorem (see Figure 2):

Theorem 1. *Let S be the point of intersection of AI and (AO). For a point T on $(AO)'$, let δ_1 be the incircle of the curvilinear triangle made by (AO), (AB) and the line IT, and let δ_2 be the circle touching $(AO)'$, $(AB)'$ and IT from the side opposite to δ_1. Then the two circles δ_1 and δ_2 are congruent if and only if $T = S'$. In this event their common radius equals*

$$\frac{4a^2b}{(2a+b)^2}.$$

The congruent circles are stated in [1], and Theorem 1 gives a necessary and sufficient condition giving the two congruent circles. In this article we generalize the theorem to a generalized arbelos called the collinear arbelos.

2. Collinear arbelos

For a point P on the half line with initial point A passing through B, let Q be the point on the line AB such that $\overrightarrow{OA} \cdot \overrightarrow{OP} = \overrightarrow{OB} \cdot \overrightarrow{OQ}$, where $\cdot$ is the inner product of the vectors. Let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$. The configuration consisting of the three semicircles is called a collinear arbelos and denoted by (α, β, γ) [2, 4, 5, 6].

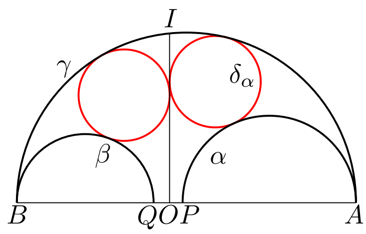

Figure 3: $-b < p < a$.

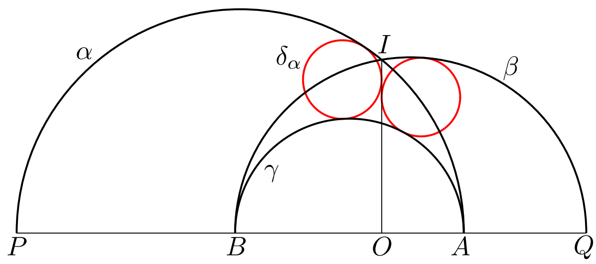

Figure 4: $p < -b$.

We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that the three semicircles lie on the region $y \geq 0$. Let $(2p, 0)$ and $(2q, 0)$ be the coordinates of the points P and Q, respectively. Notice that the axis coincides with the radical axis of α and β and the two points P and Q lie between A and B or lie in the order P, B, A, Q, which are equivalent to $-b < p < a$ and $p < -b$, respectively (see Figures 3 and 4).

Let $s = |AQ|/2$ and $t = |BP|/2$. Since $ap + bq = 0$, we have

$$ta = sb \quad and \quad tq + sp = 0. \tag{1}$$

Circles of radius $r_{\mathrm{A}} = st/(s+t)$ are called Archimedean circles of (α, β, γ). If $P = O$, then $Q = O$ and (α, β, γ) and its Archimedean circles coincide with the ordinary arbelos mentioned in the opening sentence and its Archimedean circles. Two circles in red in Figures 3 and 4 are typical Archimedean circles of (α, β, γ).

3. Generalization

In this section we generalize Theorem 1. From now on we consider a collinear arbelos (α, β, γ). We now redefine S as the point of intersection of the line AI and α.

If two congruent circles have an internal common tangent passing through the point I, one of which touches one of α and γ internally and touches the other externally, and the other circle touches one of α' and γ' internally and touches the other externally, then we call the two congruent circles an *I-congruent pair* and call the common tangent passing through I the *I-tangent*. Figure 2 shows that the circles δ_1 and δ_2 form an I-congruent pair with I-tangent IS'.

Let δ_α be the Archimedean circle touching one of α and γ internally and the other externally in Figures 3 and 4. The internal common tangents of δ_α and δ'_α meet the axis in a points closer to O than I. Hence δ_α and δ'_α do not form I-congruent pair. This implies that I-congruent pair consists of non-Archimedean circles.

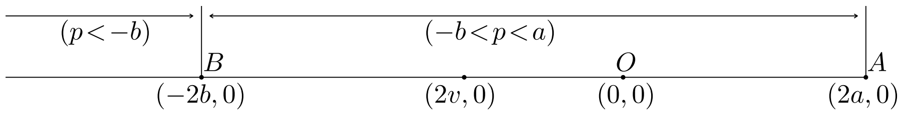

Figure 5.

Let $v = a - \sqrt{a(a+b)}$. Then $-b < v < 0$, since $v - (-b) = a + b - \sqrt{a(a+b)} > 0$. Therefore the point of coordinates $(2v, 0)$ lies between B and O (see Figure 5). This point plays an important role in considering I-congruent pairs for the collinear arbelos. Theorem 1 is generalized as follows.

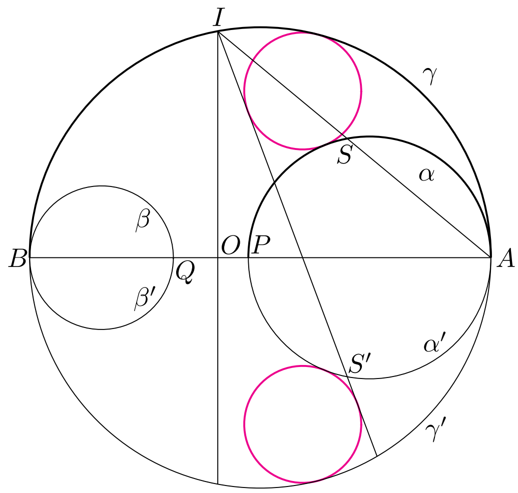

Figure 6: $v < p < a$.

Theorem 2. *The following statements are true for the collinear arbelos.*
(i) *In any case there is an I-congruent pair with I-tangent IS' of common radius*

$$\frac{4a(a-p)|b+p|}{(2a+b-p)^2}. \tag{2}$$

(ii) *If $v < p < a$, there is only one I-congruent pair stated in* (i).
(iii) *If $p = v$, there are exactly two I-congruent pairs, one is stated in* (i) *and the other consists of circles of radius*

$$\frac{2b\sqrt{a(a+b)}}{(\sqrt{a}+\sqrt{a+b})^2} \tag{3}$$

with I*-tangent* IP.
(iv) *If* $-b < p < v$ *or* $p < -b$, *there are exactly three* I*-congruent pairs associated with* I, *one of which is stated in* (i).

Proof. Let (x_s, y_s) be the coordinates of the point S. Firstly we assume $-b < p < a$. Then we get $2a : (2a - x_s) = (a+b) : (a-p)$, since A is the internal center of similitude of γ and α. Therefore by (1), we have

$$x_s = \frac{2a(b+p)}{a+b} = \frac{2at}{a+ta/s} = \frac{2st}{s+t} = 2r_{\mathrm{A}}.$$

Also by the same similarity, we have

$$y_s = 2\sqrt{ab}(a - r_{\mathrm{A}})/a = 2(a - r_{\mathrm{A}})\sqrt{\frac{b}{a}}.$$

We assume that there is an I-congruent pair of common radius r and centers of coordinates $(e, \pm f)$. Since their I-tangent passes through the midpoint of the segment joining the two centers and I has coordinate $(0, 2\sqrt{ab})$, the I-tangent has an equation

$$2\sqrt{ab}(x - e) + ey = 0. \tag{4}$$

The distances from the centers of the congruent circles to the centers of α, γ and the I-tangent equal $a - p + r$, $a + b - r$ and r, respectively. Therefore we get the following three equations.

$$(a + p - e)^2 + f^2 = (a - p + r)^2, \tag{5}$$

$$(a - b - e)^2 + f^2 = (a + b - r)^2, \tag{6}$$

$$\frac{e^2 f^2}{4ab + e^2} = r^2. \tag{7}$$

Conversely from real numbers e, f and $r > 0$ satisfying the three equations, we get an I-congruent pair with centers of coordinates $(e, \pm f)$ and common radius r. Eliminating f from (5) and (6), and also from (5) and (7), we have

$$b(e + r) + p(e - r) - 2a(b + p - r) = 0, \tag{8}$$

$$4abr^2 + e^2((e - 2a)(e - 2p) - 2(a - p)r) = 0. \tag{9}$$

It is sufficient to consider the existence of real numbers e and r satisfying (8) and (9), because f is determined by (7). Solving (8) and (9) for e and r, we get

$$e = 2a \quad \text{and} \quad r = 0, \tag{10}$$

or

$$e = \frac{2a(b+p)}{2a+b-p} \quad \text{and} \quad r = \frac{4a(a-p)(b+p)}{(2a+b-p)^2}, \tag{11}$$

or

$$e = \frac{2\left(b(p-a) \mp \sqrt{w}\right)}{2a+b-p} \quad \text{and} \quad r = \frac{2(b+p)\left((a+b)(2a-p) \pm \sqrt{w}\right)}{(2a+b-p)^2}, \tag{12}$$

where $w = b(a+b)(p^2 - a(b+2p)) = b(a+b)(p-v)(p-a-\sqrt{a(a+b)})$. Notice that $w \geq 0$ implies $r > 0$ in (12), since

$$(a+b)^2(2a-p)^2 - w = a(a+b)(2a+b-p)^2 > 0. \tag{13}$$

While $w \geq 0$ if and only if $p \leq v$ (see Figure 7). Therefore (12) gives no I-congruent pair, one I-congruent pair or two I-congruent pairs, according as $v < p < a$, $p = v$ or $-b < p < v$.

We secondly assume $p < -b$ and consider an I-congruent pair of common radius r and centers of coordinates $(e, \pm f)$. Then we get

$$(x_s, y_s) = \left(-2r_{\mathrm{A}}, 2(a + r_{\mathrm{A}})\sqrt{\frac{b}{a}}\right)$$

similarly. Since the distances from the centers of the congruent circles to the centers of α and γ equal $a - p - r$, $a + b + r$, respectively, the relations between e, f, r, a, b and p are obtained by changing the signs of r in (5), (6) and (7). Therefore we get

$$e = 2a \quad \text{and} \quad r = 0, \tag{14}$$

or

$$e = \frac{2a(b + p)}{2a + b - p} \quad \text{and} \quad r = \frac{4a(a - p)(-b - p)}{(2a + b - p)^2}, \tag{15}$$

or

$$e = \frac{2\left(b(p - a) \mp \sqrt{w}\right)}{2a + b - p} \quad \text{and} \quad r = \frac{2(b + p)\left(-(a + b)(2a - p) \mp \sqrt{w}\right)}{(2a + b - p)^2}. \tag{16}$$

Then (16) gives two I-congruent pairs, since $p < -b$ implies $w > 0$ and $r > 0$ by (13) in (16).

We now prove (i). We exclude the case given by (10) and (14) since $r = 0$, which will be considered in the next section. If we consider the I-congruent pairs given by (11) or (15), then in any case $e = 2a(b + p)/(2a + b - p)$, and we have

$$2\sqrt{ab}(x_s - e) + e(-y_s) = \frac{-8\sqrt{ab}(b + p)(ap + bq)}{(2a + b - p)(a + b + p - q)} = 0,$$

where we use $s = a - q$ and $t = p + b$ if $-b < p$, and $s = q - a$ and $t = -b - p$ if $p < -b$. Therefore the point S' lies on the I-tangent expressed by (4). This proves (i).

If $v < p < a$, we get only one I-congruent pair given by (11). This proves (ii) (see Figure 6). If $p = v$, (12) gives one I-congruent pair. Substituting $p = v$ in (12), we get

$$e = \frac{2b(p - a)}{2a + b - p} = 2(a - \sqrt{a(a + b)}) = 2p \quad \text{and} \quad r = \frac{2b\sqrt{a(a + b)}}{(\sqrt{a} + \sqrt{a + b})^2}.$$

Hence the I-tangent coincides with the line IP and the common radius is given by (3). This proves (iii) (see Figure 8). If $-b < p < v$, then (12) gives two I-congruent pairs (see Figure 9). If $p < -b$, (16) also gives two I-congruent pairs. This proves (iv) (see Figure 10). □

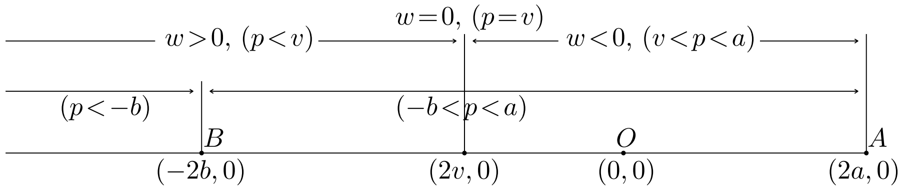

Figure 7.

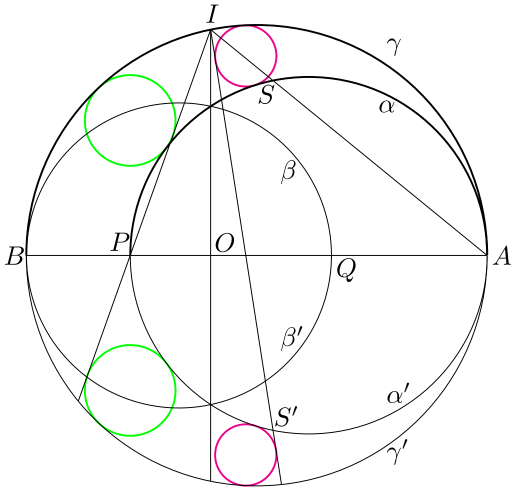

Figure 8: $p = v$.

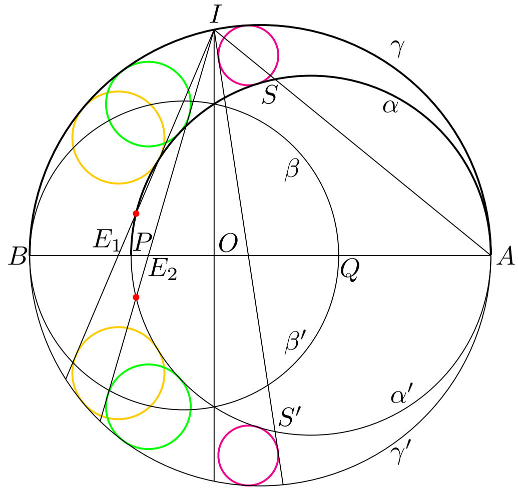

Figure 9: $-b < p < v$.

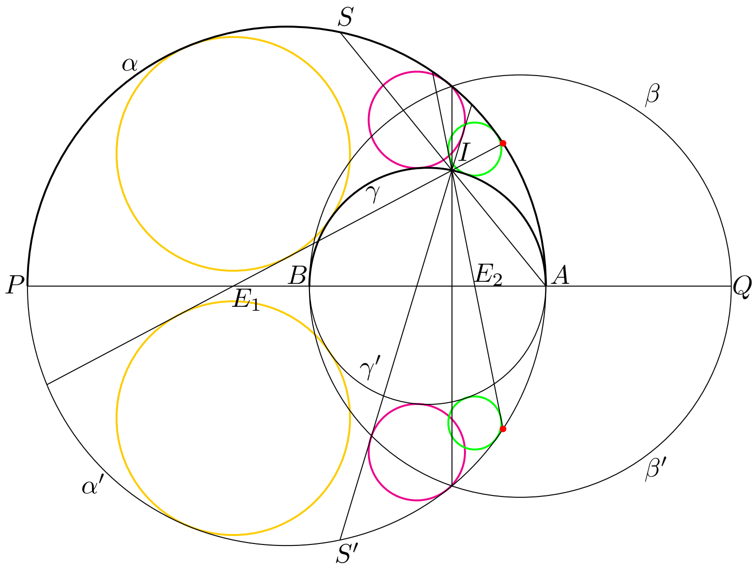

Figure 10: $p < -b$.

4. Two I-congruent pairs relevant to each other

Let us consider two I-congruent pairs. If one of the points of intersection of α and one of their I-tangents coincides with the reflection of the point of intersection of α' and the other I-tangent, the two I-congruent pairs are said to be *relevant to each other*. In Figure 2, the point A can be regarded as a trivial I-congruent pair of radius 0 with I-tangent IS, if we consider A as overlapping two point circles, which is also suggested by (10) and (14). Therefore Figure 2 shows the two I-congruent pairs with I-tangents IS and IS' relevant to each other. The trivial I-congruent pair is also relevant to itself, since IA coincides with the I-tangent and $A' = A$.

We assume the case (iv) in Theorem 2. Let E_1 and E_2 be the points of coordinates

$$\left(\frac{2(b(p-a)-\sqrt{w})}{2a+b-p}, 0\right) \text{ and } \left(\frac{2(b(p-a)+\sqrt{w})}{2a+b-p}, 0\right),$$

respectively. The lines E_1I and E_2I are the I-tangents of the two I-congruent pairs given by (12) and (16). One of the points of intersection of E_1I and α has coordinates

$$\left(-\frac{2a(b+p)}{a-p}, \frac{2\sqrt{aw/b}}{a-p}\right).$$

On the other hand, the point of intersection of E_2I and α' has coordinates

$$\left(-\frac{2a(b+p)}{a-p}, -\frac{2\sqrt{aw/b}}{a-p}\right).$$

Therefore the two I-congruent pairs are relevant to each other (see Figures 9 and 10, where the two points of intersection are colored in red). Notice that the case described in Theorem 2 (iii) is obtained if the two points E_1, E_2 and P coincide. Since $P = P'$ in this event, the I-congruent pair is relevant to itself. Including the trivial I-congruent pair and the concept of the relevancy, we restate Theorem 2 as follows.

Theorem 3. *The following statements are true for the collinear arbelos.*
(i) *There are two I-congruent pairs relevant to each other in any case. One consists of the point circle A with I-tangent IS, and the other consists of circles of radius given by* (2) *with I-tangent IS'.*
(ii) *If $v < p < a$, there are exactly two I-congruent pairs relevant to each other stated in* (i)*.*
(iii) *If $p = v$, there are exactly three I-congruent pairs, two of which are the pairs relevant to each other stated in* (i)*. The remaining I-congruent pair is relevant to itself and consists of circles of radius given by* (3) *with I-tangent IP.*
(iv) *If $-b < p < v$ or $p < -b$, there are exactly four I-congruent pairs, two of which are the pairs relevant to each other stated in* (i)*. The remaining two pairs are also relevant to each other.*

5. The point S

Theorem 2 gives a new characterization of the point S. In this section we consider properties of this point for the collinear arbelos. The point S is now described as follows:

(i) The point of intersection of AI and α.
(ii) The point on α of x-coordinate $2r_{\mathrm{A}}$ (resp. $-2r_{\mathrm{A}}$) if $-b < p$ (resp. $p < -b$).
(iii) The point of tangency of α and the external common tangent of α and β [5].
(iv) The point such that the line joining I and the reflection of this point in AB is the I-tangent of the I-congruent pair of common radius given by (2).

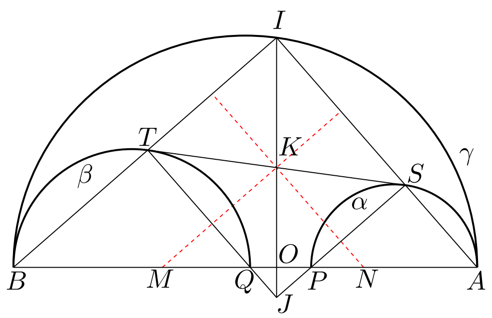

Figure 11.

Let T be the point of intersection of BI and β, and let J be the point of intersection of the lines PS and QT (see Figures 11 and 12). Then $SITJ$ is a rectangle. Since the distances from S and T to the axis are the same and equals to $2r_{\mathrm{A}}$, ST and IJ bisect each other. Let K be the midpoint of ST. Since K lies on the axis and J lies on the line IK, J also lies on the axis. If M is the midpoint of BP, then the line KM is the perpendicular bisector of IS and JT. Similarly if N is the midpoint of AQ, then KN is the perpendicular bisector of IT and JS.

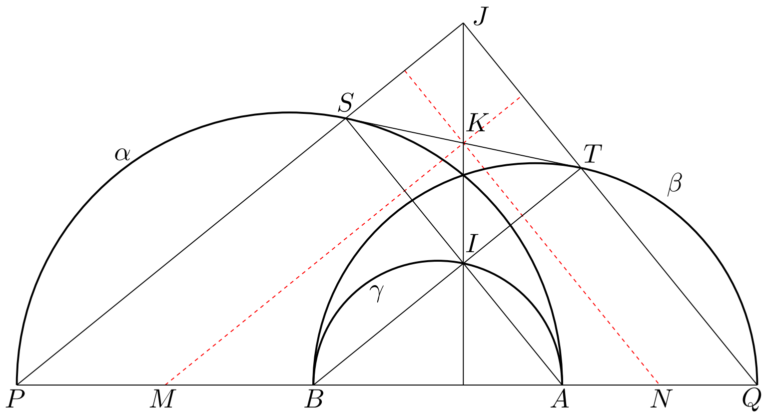

Figure 12.

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