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A note on a sangaku problem involving the Steiner ellipse

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Abstract. We generalize a sangaku problem involving the Steiner ellipse.

Keywords. sangaku, triangle, ellipse, Steiner ellipse, affine transformation.

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We consider the following problem in sangaku, which was written in 1822 on a tablet in Iwate prefecture. The tablet is now lost [1].

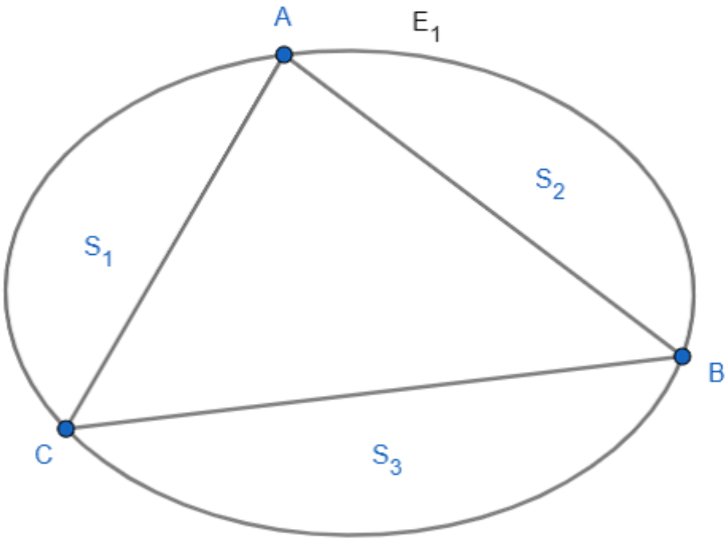

Figure 1.

Problem 1. On a given ellipse E_1, mark three points A, B and C such that the areas of ellipse segment formed by AB, BC and CA with the ellipse respectively (S_1, S_2 and S_3 in Figure 1) are equal. Show that the area of $\triangle ABC$ is

$$\frac{3\sqrt{3}ab}{4},$$

where a and b are the semimajor and semiminor axes of the ellipse, respectively.

The next theorem gives a generalization of the problem (see Figure 2).

Theorem 1. *On a given ellipse, mark n points A_1, A_2, A_3, $\cdots$, A_n such that the areas of ellipse segments formed by A_1A_2, A_2A_3 and A_3A_4, $\cdots$, $A_{n-1}A_n$, A_nA_1 with the ellipse respectively are equal. Then the area of the polygon $A_1A_2A_3\cdots A_n$ is*

$$\frac{abn}{2}\sin\left(\frac{360°}{n}\right),$$

where a and b are the semimajor and semiminor axes of the ellipse, respectively.

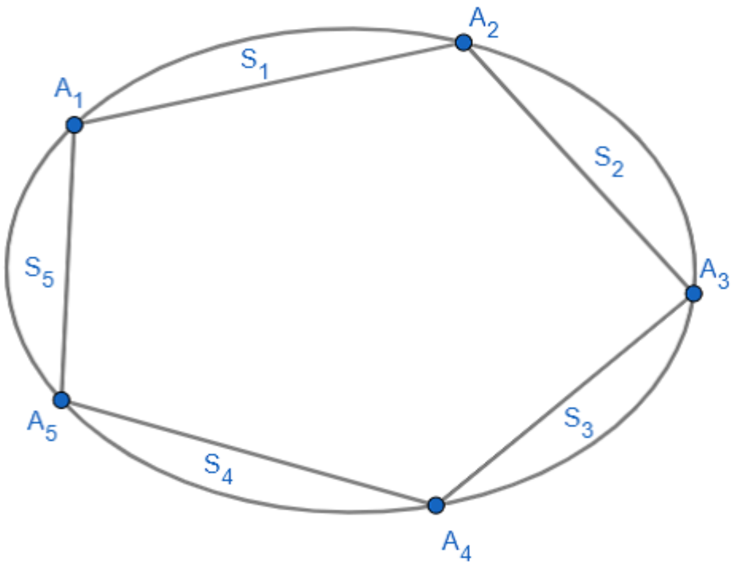

FIGURE 2. An example of $n = 5$. The areas of S_1, S_2, S_3, S_4 and S_5 are equal.

Proof. We apply an affine transformation that maps the ellipse to a unit circle, then the polygon $A_1A_2A_3\cdots A_n$ is mapped to a regular n-sided polygon. The ratio of the area of the polygon $A_1A_2A_3\cdots A_n$ to the area of the ellipse equals the ratio of the area of the regular n-sided polygon inscribed in a unit circle to the area of the unit circle. Therefore the area of $A_1A_2A_3\cdots A_n$ equals $\frac{n}{2}\sin\left(\frac{360°}{n}\right)\frac{1}{\pi}\cdot\pi ab = \frac{abn}{2}\sin\left(\frac{360°}{n}\right)$. □

Notice that E_1 is the Steiner ellipse of $\triangle ABC$, that is, its centre coincides with the centroid of $\triangle ABC$.

REFERENCES

[1] H. Fukagawa and T. Rothman, *Sacred Mathematics - Japanese Temple Geometry*, Princeton University Press, Princeton, 2008.