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A Constant Associated with Ikeda's Theorem

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Abstract. We will evaluate a constant associated with Ikeda's theorem related to the Steiner chain configuration. We will also provide a few examples from Sangaku geometry where this result can be applied.

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Mathematics Subject Classification (2010). 51M04.

1. Introduction

Let the circle ω of radius r lies inside the circle Ω of radius R. A sequence of circles $o_1, o_2, \cdots, o_n$ with the property that each circle o_i is internally tangent to Ω and externally tangent to ω, o_{i-1}, and o_{i+1} (we set $o_{n+1} = o_1$ and $o_n = o_0$) is called a *n-Steiner chain* of circles of ω and Ω (Figure 1).

The following result is known as Ikeda's theorem in Wasan geometry.

Theorem 1.1. *If $o_1, o_2, \cdots, o_{2n}$ is $2n$-Steiner chain of circles of ω and Ω of radii $r_1, r_2, \cdots, r_{2n}$, then the value of*

$$\frac{1}{r_i} + \frac{1}{r_{i+n}}$$

is constant for $i = 1, 2, \cdots, n$.

A generalization of the above theorem is given in [3] as follows.

Theorem 1.2. *If $o_1, o_2, \cdots, o_{mn}$ $(m, n \geq 2)$ is mn-Steiner chain of circles of ω and Ω of radii $r_1, r_2, \cdots, r_{mn}$, then the value of*

$$\frac{1}{r_i} + \frac{1}{r_{i+n}} + \cdots + \frac{1}{r_{i+(m-1)n}}$$

is constant for $i = 1, 2, \cdots, n$.

We will evaluate the value of the above sum in terms of known parameters.

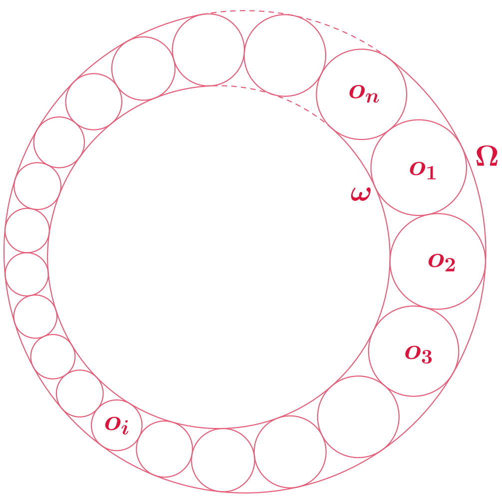

Figure 1.

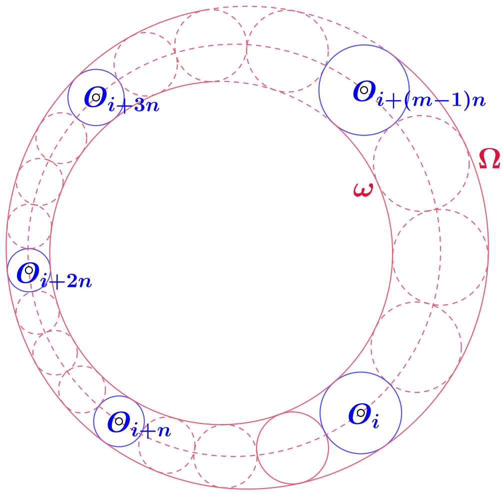

Figure 2.

2. Evaluation of the constant

We will prove the following result.

Theorem 2.1. *If* $o_1, o_2, \cdots, o_{mn}$ $(m, n \geq 2)$ *is mn-Steiner chain of circles of* ω *and* Ω *of radii* $r_1, r_2, \cdots, r_{mn}$, *then for* $i = 1, 2, \cdots, n$,

$$\frac{1}{r_i} + \frac{1}{r_{i+n}} + \cdots + \frac{1}{r_{i+(m-1)n}} = \frac{m}{2}\left(\frac{1}{R} - \frac{1}{r}\right)\cot^2\frac{\pi}{mn}.$$

Proof. We use the next theorem due to Leibniz [1].

Leibniz theorem. If $\{A_1, A_2, \cdots, A_m\}$ is a set of m distinct points with centroid G, the following relation holds for every point M in the plane:

$$\sum_{i=1}^{m} MA_i^2 = mMG^2 + \sum_{i=1}^{m} GA_i^2.$$

If the points A_1, A_2, $\cdots$, A_m form a regular m-gon of circumradius a, then G coincides with its center and the above result can be written as

$$\sum_{i=1}^{m} MA_i^2 = m(MG^2 + a^2). \tag{1}$$

It is known that by a proper choice of the center of inversion we can invert Ω and ω into a pair of concentric circles, say Ω^* and ω^* with radii R^* and r^*, respectively with common center C. Here the center of inversion S lies outside of the circles Ω. Then the circles o_1, o_2, $\cdots$, o_{mn} are mapped into circles o_1^*, o_2^*, $\cdots$, o_{mn}^* with equal radii ρ^*. Denote O_k the center of the circle o_k and O'_k the center of the circle o_k^* (see Figures 2 and 3). Let $2\theta = \angle O'_t C O'_{t+1} = 2\pi/(mn)$ (see Figure 4). From the right triangle $\triangle CDO'_t$, we have the next two equtions:

$$\sin\theta = \frac{\rho^*}{\rho^* + r^*}, \tag{2}$$

$$\rho^* = \frac{R^* - r^*}{2}. \tag{3}$$

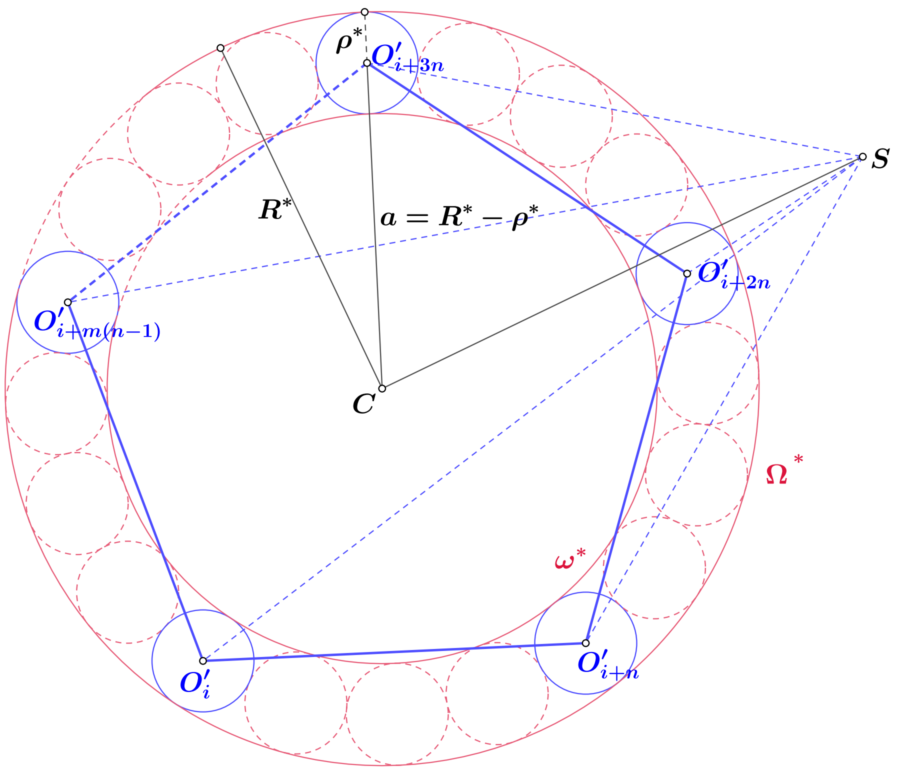

Figure 3.

Using (2) and (3), we obtain the following two relations

$$\frac{r^*}{\rho^*} = \operatorname{cosec}\theta - 1, \quad \frac{R^*}{\rho^*} = \operatorname{cosec}\theta + 1. \tag{4}$$

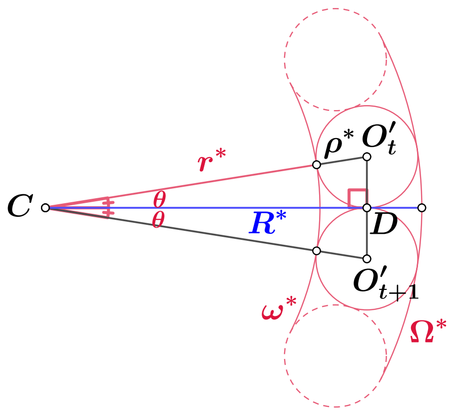

Figure 4.

Let μ be the radius of the circle of inversion. We can write

$$\frac{r^*}{r} = \frac{SC^2 - (r^*)^2}{\mu^2}, \quad \frac{R^*}{R} = \frac{SC^2 - (R^*)^2}{\mu^2}, \tag{5}$$

and for $k = 1, 2, \cdots, mn$,

$$\frac{\rho^*}{r_k} = \frac{(SO'_k)^2 - (\rho^*)^2}{\mu^2}. \tag{6}$$

Using (5), we obtain

$$SC^2 = \frac{1}{2}\left(\frac{r^*}{r} + \frac{R^*}{R} + \frac{(r^*)^2}{\mu^2} + \frac{(R^*)^2}{\mu^2}\right)\mu^2, \tag{7}$$

and

$$\frac{1}{\mu^2} = \left(\frac{1}{(R^*)^2 - (r^*)^2}\right)\left(\frac{r^*}{r} - \frac{R^*}{R}\right). \tag{8}$$

Since the points O'_i, O'_{i+n}, $\cdots$, $O'_{i+m(n-1)}$ form a regular m-gon, we get

$$(SO'_i)^2 + (SO'_{i+n})^2 + \cdots + (SO'_{i+m(n-1)})^2 = m(R^* - \rho^*)^2 + mSC^2 \tag{9}$$

by (1). Assume the required sum equals to E_i. Using (6) and (9), we can write

$$\begin{aligned} E_i =& \frac{1}{r_i} + \frac{1}{r_{i+n}} + \cdots \frac{1}{r_{i+(m-1)n}} \\ =& \frac{1}{\rho^*\mu^2}\left[(SO'_i)^2 + (SO'_{i+n})^2 + \cdots + (SO'_{i+m(n-1)})^2 - m(\rho^*)^2\right] \\ =& \frac{1}{\rho^*\mu^2}\left[m(R^* - \rho^*)^2 + mSC^2 - m(\rho^*)^2\right]. \end{aligned}$$

Simplifying the above expression using (3) and (7), we obtain

$$\begin{aligned} E_i =& \frac{1}{\rho^*\mu^2}\left[m\left(\frac{R^* + r^*}{2}\right)^2 - m\left(\frac{R^* - r^*}{2}\right)^2 + mSC^2\right] \\ =& \frac{1}{\rho^*\mu^2}\left[mR^*r^* + \frac{m\mu^2}{2}\left(\frac{r^*}{r} + \frac{R^*}{R} + \frac{(r^*)^2}{\mu^2} + \frac{(R^*)^2}{\mu^2}\right)\right] \\ =& \frac{m}{2}\left[\frac{(R^* + r^*)^2}{\rho^*\mu^2} + \frac{1}{r}\left(\frac{r^*}{\rho^*}\right) + \frac{1}{R}\left(\frac{R^*}{\rho^*}\right)\right]. \end{aligned}$$

Using (8) and multiplying the numerator and denominator of the above expression by $2\rho^* = R^* - r^*$, we get

$$E_i = \frac{m}{2}\left[\frac{(R^* + r^*)}{2(\rho^*)^2} \cdot \left(\frac{r^*}{r} - \frac{R^*}{R}\right) + \left(\frac{r^*(R^* - r^*)}{2r(\rho^*)^2}\right) + \left(\frac{R^*(R^* - r^*)}{2R(\rho^*)^2}\right)\right],$$

which gives

$$E_i = \frac{m}{2}\left(\frac{1}{r} - \frac{1}{R}\right)\frac{R^*}{\rho^*}\frac{r^*}{\rho^*}.$$

Therefore by (4) and $\theta = \pi/(mn)$, we have

$$E_i = \frac{m}{2}\left(\frac{1}{r} - \frac{1}{R}\right)(\operatorname{cosec}^2\theta - 1) = \frac{m}{2}\left(\frac{1}{r} - \frac{1}{R}\right)\cot^2\frac{\pi}{mn}. \quad \square$$

3. applications

Now we will solve two sangaku geometry problems using the above theorems.

Problem 3.1. *If o_1, o_2, $\cdots$, o_6 is 6-Steiner chain of circles of ω and Ω of radii r_1, r_2, $\cdots$, r_6, then find r_5 in terms of r_1, r_2, and r_3.*

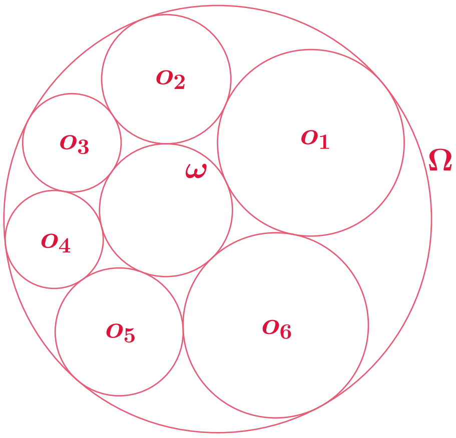

FIGURE 5.

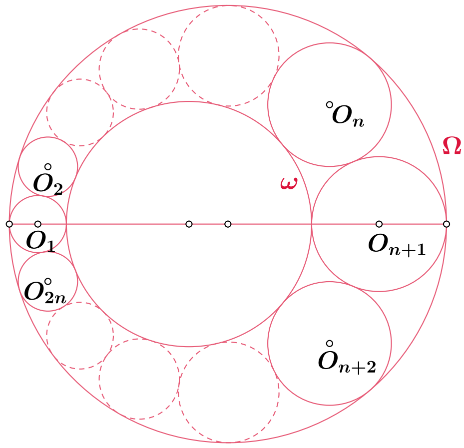

FIGURE 6.

Solution. By Theorem 1.2 for $(m,n) = (2,3)$ and $(m,n) = (3,2)$, we have the following equations:

$$\frac{1}{r_1}+\frac{1}{r_4}=\frac{1}{r_2}+\frac{1}{r_5}=\frac{1}{r_3}+\frac{1}{r_6}, \quad \frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5}=\frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

Eliminating r_4 and r_6 from the equations, we get

$$\frac{2}{r_1}+\frac{2}{r_3}=\frac{3}{r_2}+\frac{1}{r_5}, \quad \text{or} \quad r_5=\frac{r_1r_2r_3}{2r_1r_2+2r_2r_3-3r_3r_1}.$$

Problem 3.2. *o_1, o_2, $\cdots$, o_{2n} is $2n$-Steiner chain of circles of ω and Ω, where o_i has radius r_i and center O_i. If the centers of Ω and ω and the points O_1 and O_{n+1} are collinear, then find r_{n+1} in terms of R and r_1.*

Solution. By Theorem 2.1 for $m = 2$ and $i = 1$ we have

$$\frac{1}{r_1}+\frac{1}{r_{n+1}}=\left(\frac{1}{r}-\frac{1}{R}\right)\cot^2\frac{\pi}{2n}. \tag{10}$$

Since O_1, O_{n+1} and the centers of Ω and ω are collinear (Figure 6), we have

$$r = R - r_1 - r_{n+1}. \tag{11}$$

Substituting (11) in (10) and solving the resulting equation for r_{n+1}, we have

$$r_{n+1}=\frac{R(R-r_1)}{R+r_1\cot^2\dfrac{\pi}{2n}}.$$

Problems 3.1 and 3.2 can be found in [2] with no solution but the final answers. However surprisingly both the answers are incorrect!

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