Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 1 (2017), pp. 24–34
Received 24 September 2017. Published on-line 16 October 2017
web: http://www.sangaku-journal.eu/

Configurations of congruent circles on a line

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Abstract. Problems involving several congruent circles on a line are considered, which yields several configurations of congruent circles on a line.

Keywords. congruent circles on a line

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction and Preliminaries

Let α_1, α_2, $\cdots$, α_n be congruent circles touching a line s from the same side such that α_1 and α_2 touch, and α_i $(i = 3, 4, \cdots, n)$ touches α_{i-1} from the side opposite to α_1. We call α_1, α_2, $\cdots$, α_n congruent circles on a line or congruent circles on s (see Figure 1). In this paper we consider several problems involving congruent circles on a line and construct several configurations consisting of congruent circles on a line. If each of α, β, γ is a line or a circle and they form a curvilinear triangle, we denote the triangle and its incircle by $T(\alpha, \beta, \gamma)$ and $I(\alpha, \beta, \gamma)$, respectively. If one of α, β, γ is a circle and the others are tangents of the circle parallel to each other, $I(\alpha, \beta, \gamma)$ is one of the two circles congruent to the circle touching the three (see Figure 2). We use the following propositions.

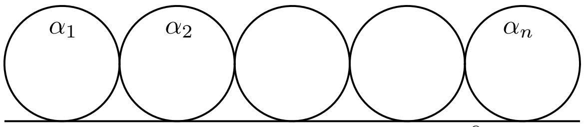

Figure 1: $n = 5$

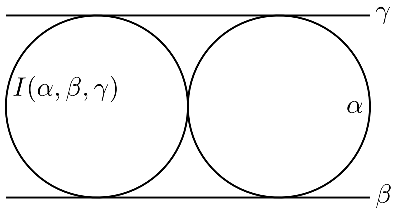

Figure 2.

Proposition 1.1. *If α and β are externally touching circles of radii a and b with external common tangent s, the following statements hold.*

(i) *If s touches the two circles at points P and Q, $|PQ| = 2\sqrt{ab}$.*
(ii) *If c is the radius of $I(\alpha, \beta, s)$, then the following relation holds:*

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}. \tag{1}$$

A sangaku problem dated 1824 in Gunma is sometimes cited for Proposition 1.1(ii) [3], [4], but the same problem can be found in several older books [1], [9], [10], [11], [15], [17], [18], where the original of [17] was written in 1796 [8].

Proposition 1.2. *Let α, β, γ be circles of radii a, b, c, respectively. If s and t are tangents of β parallel to each other, α touches s from the same side as β and β externally, and γ touches t from the same side as β and α and β externally, the following relation holds:*

$$c = \frac{b^2}{4a}. \tag{2}$$

Proof. Let A and C be the centers of α and γ, respectively, and let F be the foot of perpendicular from C to the line parallel to s passing through A (see Figure 3). We get $|AF| = |2\sqrt{ab} - 2\sqrt{bc}|$ by Proposition 1.1(i), also $|CF| = |a - 2b + c|$ and $|AC| = a + c$. Solving the equation $|AF|^2 + |CF|^2 = |AC|^2$ for c, we get (2). □

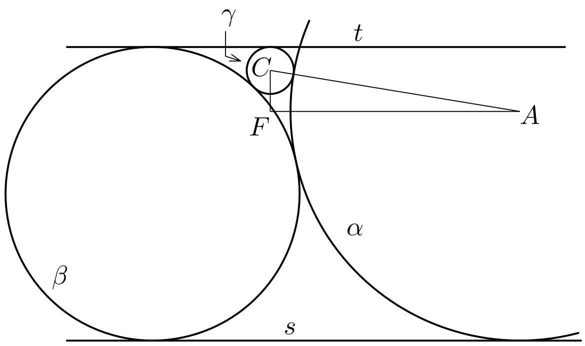

Figure 3.

Proposition 1.2 can be found in [1] and [3], where the condition $b > a$ is assumed. But such an assumption is unnecessary.

2. Problems involving congruent circles on a line

If β_1, β_2, $\cdots$, β_n ($n \geq 2$) are congruent circles on a line s, and a circle α touches β_1, β_n and s, we denote the configuration consisting of α, β_1, β_2, $\cdots$, β_n and s by $\mathcal{A}(n)$ (see Figure 4). If a circle α touches a line s and β_1 is the remaining tangent of α parallel to s, the configuration consisting of α, β_1 and s is denoted by $\mathcal{A}(1)$ (see Figure 5). We call α and s the center circle and the baseline of $\mathcal{A}(n)$. The circles β_1 and β_n (if $n \geq 2$) are called the sides of $\mathcal{A}(n)$. If $n \geq 2$, the remaining tangent of β_1 parallel to s is called the auxiliary line of $\mathcal{A}(n)$.

There are several problems involving $\mathcal{A}(n)$ especially in the case $n = 4, 5$ in Wasan geometry. For the case $n = 5$, a problem proposed by Shinohara (篠原善成) dated 1809 with Figure 6 can be found in [16]. Also $\mathcal{A}(5)$ can be found in several problems [7], [12], [14], [21], [22], where the problem in [7] is using a figure arranged as in Figure 7. Problems involving $\mathcal{A}(4)$ can be found in [2], [6], [16], [19], [20], [21]. Problems involving $\mathcal{A}(2)$ can be found in [5], [21], [22]. All the problems are essentially asking to find the ratio of the two different radii of the circles forming $\mathcal{A}(n)$. The next theorem gives a general solution of those problems.

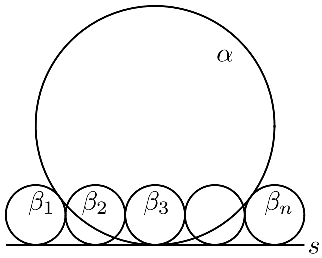

Figure 4: $\mathcal{A}(n)$, $n = 5$

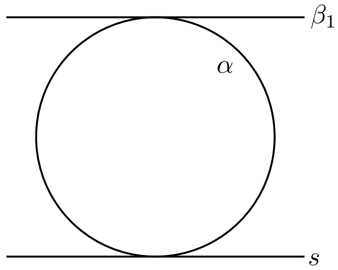

Figure 5: $\mathcal{A}(1)$

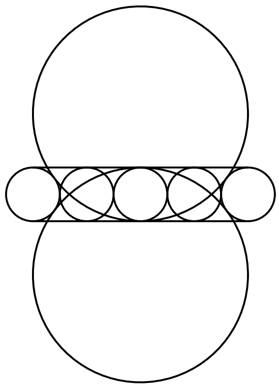

Figure 6.

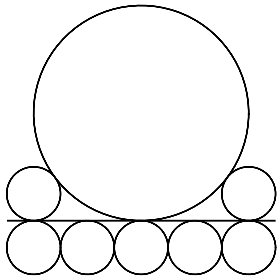

Figure 7.

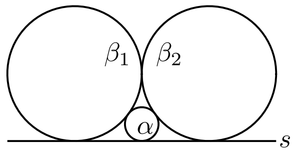

Figure 8: $\mathcal{A}(2)$

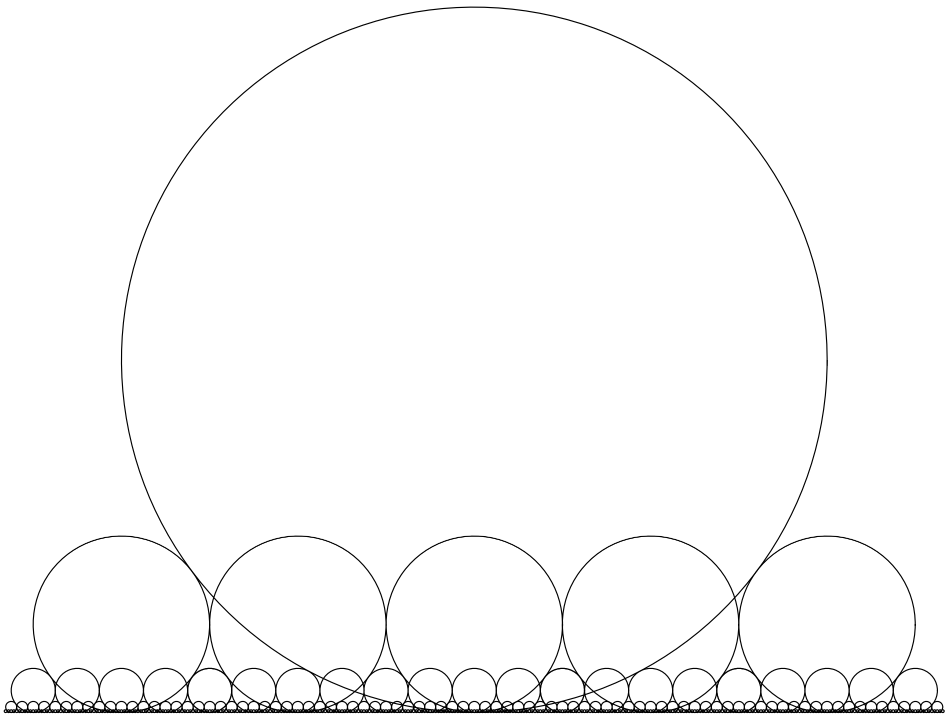

Figure 9.

Theorem 2.1 ([13])**.** *If the center circle and the sides of* $\mathcal{A}(n)$ $(n \geq 2)$ *have radii* a *and* b*, respectively, the following equation holds.*

$$\frac{a}{b} = \left(\frac{n-1}{2}\right)^2. \tag{3}$$

Remark 1. If we do not distinguish similar figures, the figure satisfying (3) is uniquely determined. Hence the converse of the theorem is also true.

Since $a/b = 1/4$ in $\mathcal{A}(2)$ (see Figure 8), and $a/b = 4$ in $\mathcal{A}(5)$, we can construct a recursive configuration denoted by Figure 9. Figure 10 is made by using Shinohara's figure, where the horizontal parallel segments are removed (see Figure 6).

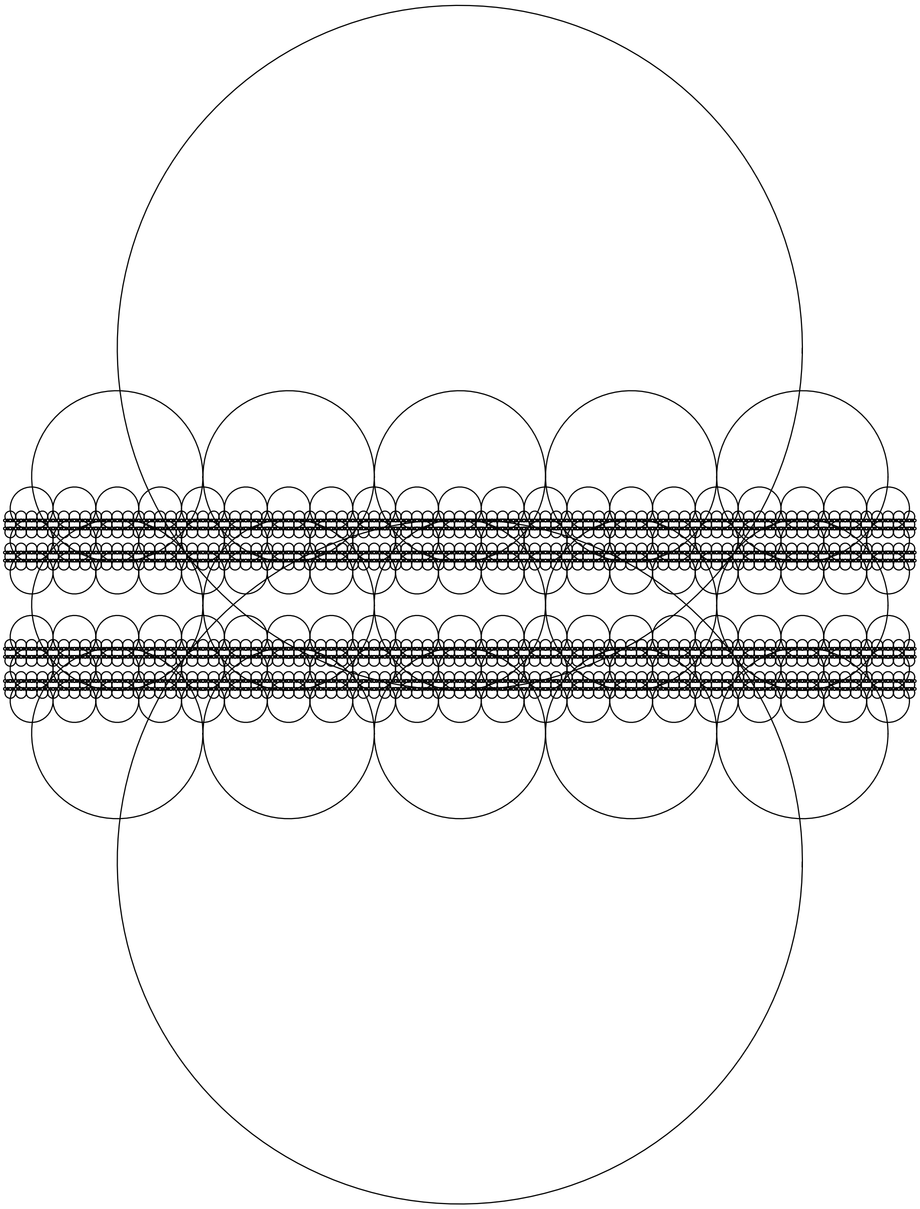

Figure 10.

Let M be the midpoint of the segment joining the centers of the sides of $\mathcal{A}(n)$ $(n \geq 2)$ with center circle α, and let A and B be the centers of α and one of the sides. Then $|AM| : |BM| = (n+1)|n-3| : 4(n-1)$. Therefore ABM is a $3:4:5$ triangle if and only if $n = 2, 5, 7$. And ABM is a $5:12:13$ triangle if and only if $n = 4, 11$. But there is no natural number n such that ABM is a $555:572:797$ triangle.

3. SOME PROPERTIES OF $\mathcal{A}(n)$

In this section we consider properties of $\mathcal{A}(n)$.

Theorem 3.1. *If $\mathcal{A}(m)$ $(m \geq 2)$ has center circle α and one of the sides β, $\mathcal{A}(n)$ $(n \geq 2)$ has center circle β and one of the sides γ, and m or n is odd, then α and congruent circles on a line congruent to γ form*

$$\mathcal{A}\left(\frac{(m-1)(n-1)}{2}+1\right).$$

Proof. Since the ratio of the two different radii of the circles forming $\mathcal{A}(m)$ equals $((m-1)/2)^2 : 1$, the ratio of the radii of α and γ is $((m-1)(n-1)/4)^2 : 1$. While solving the equation $((m-1)(n-1)/4)^2 = ((x-1)/2)^2$ for positive number x, we get $x = (m-1)(n-1)/2+1$. Hence the theorem is proved by Remark 1. □

Theorem 3.2. *If α is the center circle of $\mathcal{A}(n)$ $(n \geq 1)$ with one of the sides β and baseline s, $I(\alpha, \beta, s)$ is one of the sides of $\mathcal{A}(n+2)$ with center circle α baseline s.*

Proof. The theorem is obvious if $n = 1$. Let $n \geq 2$. Let a, b, c be the radii of α, β, $I(\alpha, \beta, s)$, respectively. Then from (1) and (3) we have

$$c = \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2} = \frac{a}{\left(\sqrt{a/b}+1\right)^2} = \frac{a}{((n+1)/2)^2}.$$

Hence the theorem follows from $a/c = (((n+2)-1)/2)^2$ by Remark 1. □

Theorem 3.3. *If α is the center circle of $\mathcal{A}(n)$ $(n \geq 3)$ with one of the sides β and auxiliary line t, $I(\alpha, \beta, t)$ is one of the sides of $\mathcal{A}(2n-1)$ with center circle β baseline t.*

Proof. Let a, b, c be the radii of α, β, $I(\alpha, \beta, t)$, respectively (see Figure 11). Since (2) and (3) hold, the theorem follows from

$$\frac{b}{c} = \frac{4a}{b} = (n-1)^2 = \left(\frac{(2n-1)-1}{2}\right)^2.$$

□

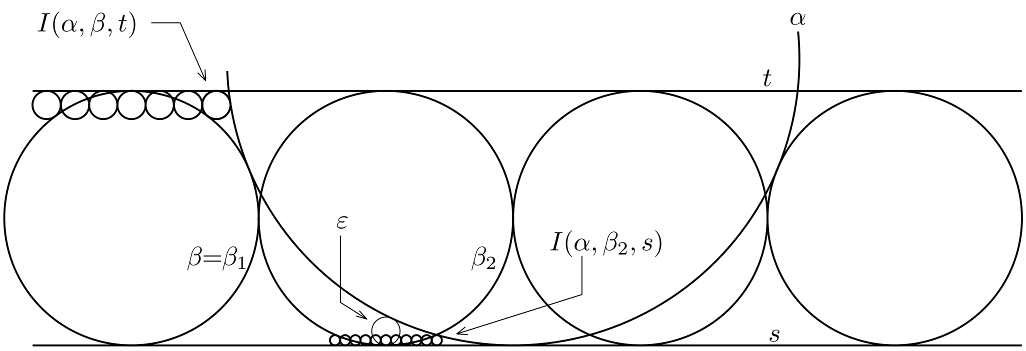

Figure 11.

Remark 2. If α is the center circle of $\mathcal{A}(2)$ with sides $\beta = \beta_1$ and β_2 and auxiliary line t, the theorem still holds in the case $n = 2$ if we define $I(\alpha, \beta_1, t) = \beta_2$.

Theorem 3.4. *If congruent circles $\beta_1, \beta_2, \cdots, \beta_n, \cdots, \beta_{2n}$ $(n \geq 1)$ on a line form $\mathcal{A}(2n)$ with center circle α baseline s and auxiliary line t, the followings hold.*
(i) *$I(\alpha, \beta_n, s)$ is one of the sides of $\mathcal{A}(4n+3)$ with center circle β_n baseline s.*
(ii) *If $n \geq 2$ and ε is the circle touching α externally and s at the point of tangency of β_n and s, then the circles ε and $I(\alpha, \beta_1, t)$ are congruent.*

Proof. Let a, b, c be the radii of α, β_n, $I(\alpha, \beta_n, s)$, respectively (see Figure 11). By Proposition 1.1(i) we have $2\sqrt{ac}+2\sqrt{bc} = b$, i.e., $c = b^2/(2(\sqrt{a}+\sqrt{b}))^2$. Since $\sqrt{a/b} = (2n-1)/2$, the part (i) follows from

$$\frac{b}{c} = \frac{4(\sqrt{a}+\sqrt{b})^2}{b} = 4\left(\sqrt{\frac{a}{b}}+1\right)^2 = (2n+1)^2 = \left(\frac{(4n+3)-1}{2}\right)^2.$$

If e is the radius of ε, then $2\sqrt{ae} = b$. Hence (ii) follows from Proposition 1.2. □

Theorem 3.4(ii) holds in the case $n = 1$ if we define $I(\alpha, \beta_1, t)$ as in Remark 2.

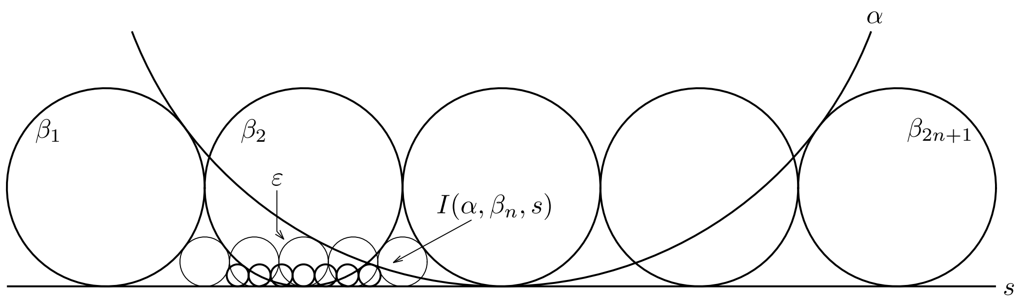

Figure 12: $n = 2$

Theorem 3.5. *If congruent circles $\beta_1, \beta_2, \cdots, \beta_{2n+1}$ $(n \geq 1)$ on a line form $\mathcal{A}(2n+1)$ with center circle α baseline s, the following statements hold.*
(i) *$I(\alpha, \beta_n, s)$ is one of the sides of $\mathcal{A}(2n+3)$ with center circle β_n baseline s.*
(ii) *If ε is the circle touching α externally and s at the point of tangency of β_n and s, ε is a member of the congruent circles on s forming $\mathcal{A}(2n+1)$ with center circle β_n.*

Proof. Let a, b, c be the radii of α, β_n, $I(\alpha, \beta_n, s)$, respectively (see Figure 12). From $2\sqrt{bc}+2\sqrt{ac} = 2b$, $c = b^2/(\sqrt{a}+\sqrt{b})^2$. Since $\sqrt{a/b} = n$, (i) follows from

$$\frac{b}{c} = \frac{\left(\sqrt{a}+\sqrt{b}\right)^2}{b} = \left(\sqrt{\frac{a}{b}}+1\right)^2 = (n+1)^2 = \left(\frac{(2n+3)-1}{2}\right)^2.$$

If e is the radius of ε, $2\sqrt{ae} = 2b$, i.e., $b/e = a/b$. This proves (ii). □

4. Configurations consisting of $\mathcal{A}(n)$

In this section we construct configurations consisting of $\mathcal{A}(n)$. Let β_1^1 be the line forming $\mathcal{A}(1)$ with center circle α and baseline s. If congruent circles $\beta_1^k, \beta_2^k, \cdots, \beta_k^k$ on a line form $\mathcal{A}(k)$ with center circle α one of the sides β_1^k and baseline s for $k = 2n-1$, let $\beta_1^{k+2} = I(\alpha, \beta_1^k, s)$. Then β_1^{k+2} is one of the sides of $\mathcal{A}(k+2)$ with center circle α baseline s by Theorem 3.2. Hence by induction we get a configuration consisting of $\mathcal{A}(1)$, $\mathcal{A}(3)$, $\mathcal{A}(5)$, $\cdots$, $\mathcal{A}(2n-1)$, $\cdots$ with common center circle α and common baseline s. The configuration is denoted by $\mathcal{A}_o$, and α and s are also called the center circle and the baseline of $\mathcal{A}_o$ (see Figure 13). Since

β_1^1, β_1^3, β_1^5, $\cdots$ form a chain of circles touching α and s, $\mathcal{A}_o$ can be constructed from α, s and one of the circles in the chain.

Similarly starting with $\mathcal{A}(2)$ with center circle α and baseline s, we get a configuration consisting of $\mathcal{A}(2)$, $\mathcal{A}(4)$, $\mathcal{A}(6)$, $\cdots$, $\mathcal{A}(2n)$, $\cdots$ with common center circle α and common baseline s. The configuration is denoted by $\mathcal{A}_e$, and α and s are also called the center circle and the baseline of $\mathcal{A}_e$ (see Figure 14). Circles touching α in $\mathcal{A}_e$ form a chain of circles touching α and s. Therefore $\mathcal{A}_e$ can also be constructed from α, s and one of the circles in the chain.

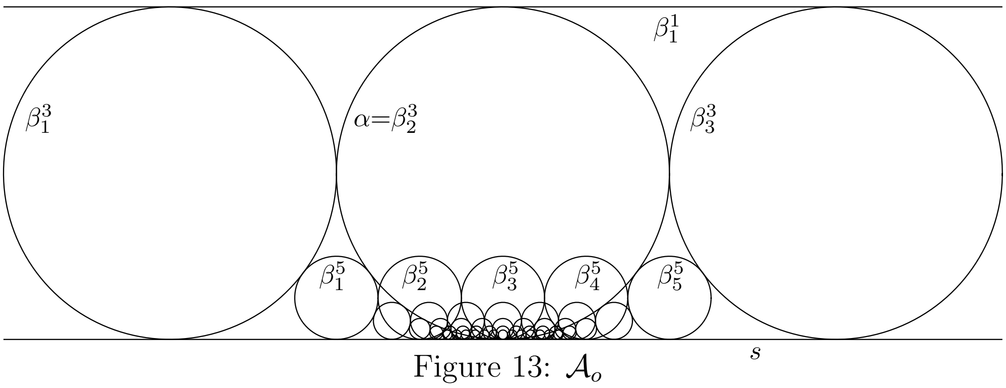

Figure 13: $\mathcal{A}_o$

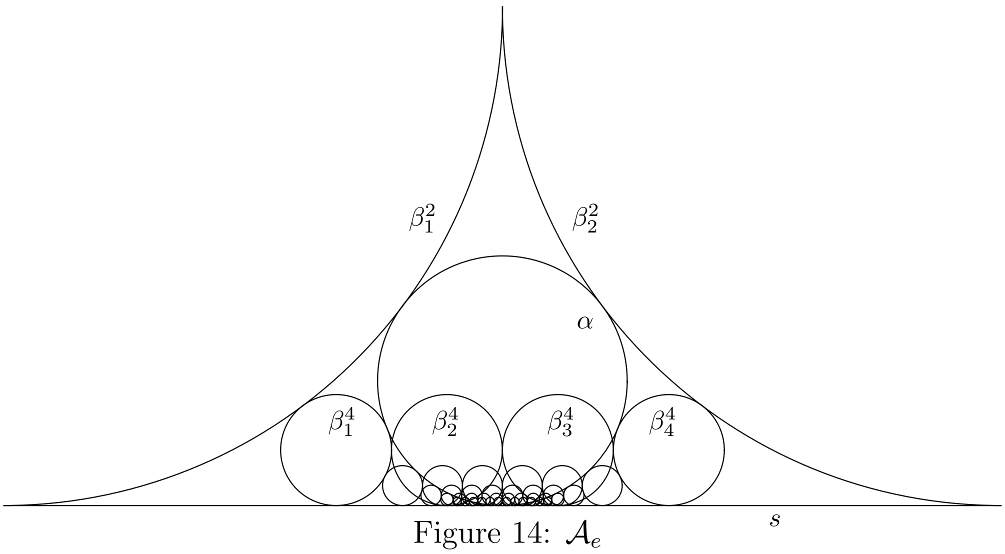

Figure 14: $\mathcal{A}_e$

5. ANOTHER CONFIGURATIONS OF CONGRUENT CIRCLES ON A LINE

Let β_1 and β_2 be congruent touching circles with external common tangent s. Let γ_1, γ_2, $\cdots$, γ_n be congruent circles on s such that they lie in the curvilinear triangle $T(\beta_1, \beta_2, s)$, γ_1 touches β_1 and γ_n touches β_2. The configuration consisting of β_1, β_2, γ_1, γ_2, $\cdots$, γ_n, and s is denoted by $\mathcal{B}(n)$ (see Figure 15). The two circles β_1 and β_2 and the line s are called the sides and the baseline of $\mathcal{B}(n)$, and γ_1 and γ_n (if $n \geq 2$) are called the inner sides of $\mathcal{B}(n)$. The two configurations $\mathcal{B}(1)$ and $\mathcal{A}(2)$ are the the same. A problem involving $\mathcal{B}(5)$ can be found in [20].

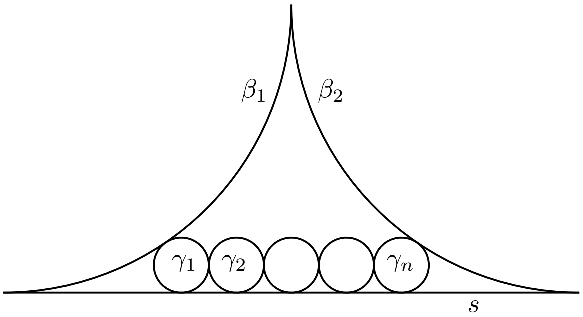

Figure 15: $\mathcal{B}(n)$, $n = 5$

Theorem 5.1. *The following statements are true for $\mathcal{B}(n)$.*
(i) *Circles of radii b and c ($b > c$) can form $\mathcal{B}(n)$ if and only if*

$$\frac{b}{c} = \left(\sqrt{n} + 1\right)^2.$$

(ii) *If β_1 is one of the sides of $\mathcal{B}(n^2)$ with baseline s and γ_1 is one of the the inner sides of $\mathcal{B}(n^2)$ touching β_1, $I(\beta_1, \gamma_1, s)$ is one of the inner sides of $\mathcal{B}((n+1)^2)$ with one of the sides β_1 baseline s.*

Proof. If circles of radii b and c ($b > c$) form $\mathcal{B}(n)$ with baseline s, the distance between the points of tangency of the sides and s equals $2(n-1)c + 4\sqrt{bc} = 2b$. This gives the equation in (i). The converse holds by the uniqueness of the figure. This proves (i). Let b and c be the radii of β_1 and γ_1 forming $\mathcal{B}(n^2)$, respectively, and let d be the radius of $I(\beta_1, \gamma_1, s)$. Then $d = bc/(\sqrt{b} + \sqrt{c})^2 = b/(\sqrt{b/c} + 1)^2 = b/(n+2)^2$ by Proposition 1.1(ii) and (i). Hence (ii) is proved by $b/d = \left(\sqrt{(n+1)^2} + 1\right)^2$. □

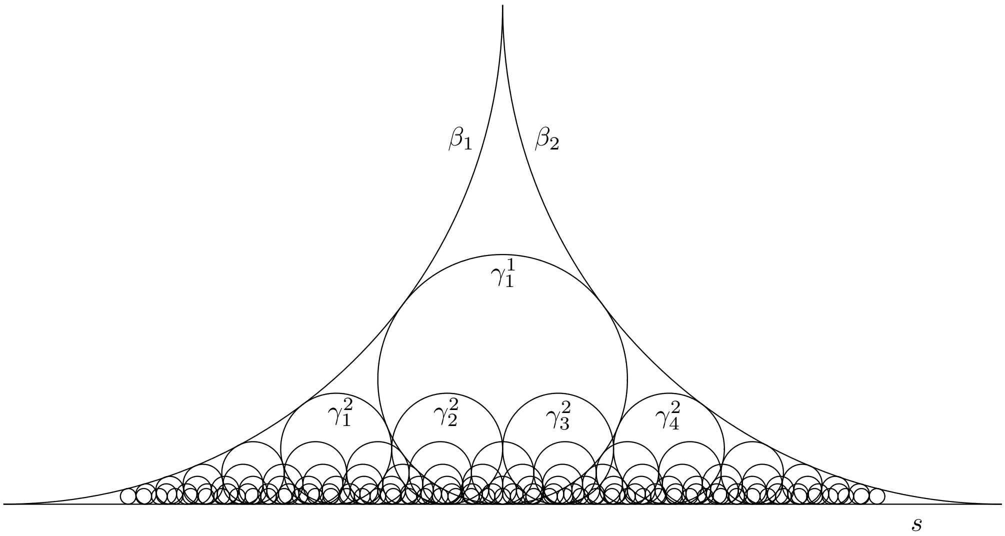

Figure 16: $\mathcal{B}$

Let us assume that a circle γ_1^1 forms $\mathcal{B}(1)$ with sides β_1 and β_2 baseline s. If congruent circles $\gamma_1^k, \gamma_2^k, \cdots, \gamma_{k^2}^k$ on a line form $\mathcal{B}(k^2)$ with sides β_1 and β_2 baseline s, where γ_1^k touches β_1, then $I(\beta_1, \gamma_1^k, s)$ is one of the inner sides of $\mathcal{B}((k+1)^2)$ with sides β_1 and β_2 baseline s by Theorem 5.1(ii). Hence we get a configuration consisting of $\mathcal{B}(1^2)$, $\mathcal{B}(2^2)$, $\mathcal{B}(3^2)$, $\cdots$ with common sides β_1 and β_2 and common baseline s by induction. The configuration is denoted by $\mathcal{B}$, and β_1 and β_2 and s are also called the sides and the baseline of $\mathcal{B}$ (see Figure 16). The circles touching β_1 in $\mathcal{B}$ form a chain of circles touching β_1 and s. Therefore $\mathcal{B}$ can be constructed from β_1, s and one of the circles in the chain.

Theorem 5.2. *If γ_1^n, γ_2^n, $\cdots$, $\gamma_{n^2}^n$ $(n = 1, 2, \cdots)$ are congruent circles on a line forming $\mathcal{B}(n^2)$ in $\mathcal{B}$ with baseline s, the following statements hold.*
(i) *If $j = n(n-1)/2$ and $n \geq 2$, then γ_j^n and γ_{j+n+1}^n are the sides of $\mathcal{A}(n+2)$ with center circle γ_1^1 baseline s.*
(ii) *The configurations $\mathcal{A}(2)$ and $\mathcal{A}(4)$, $\mathcal{A}(5)$, $\mathcal{A}(6)$, $\cdots$ with center circle γ_1^1 baseline s are contained in $\mathcal{B}$.*
(iii) *$\mathcal{A}_e$ with center circle γ_1^1 baseline s is contained in $\mathcal{B}$.*

Proof. Let a, b, c be the radii of γ_1^1, the sides of $\mathcal{B}$, γ_1^n $(n \geq 2)$, respectively. Since $b/a = 4$ and $b/c = (n+1)^2$, $a/c = (n+2-1)^2/4$. Hence circles congruent to γ_1^n can form $\mathcal{A}(n+2)$ with center circle γ_1^1 by Remark 1. While $2\sqrt{bc}+2(j-1)c+2\sqrt{ac} = 2(n+1)c + 2(n(n-1)/2-1)c + (n+1)c = (n+1)^2c = b$ shows that γ_j^n touches γ_1^1 externally. This proves (i). The parts (ii) and (iii) follow from (i). □

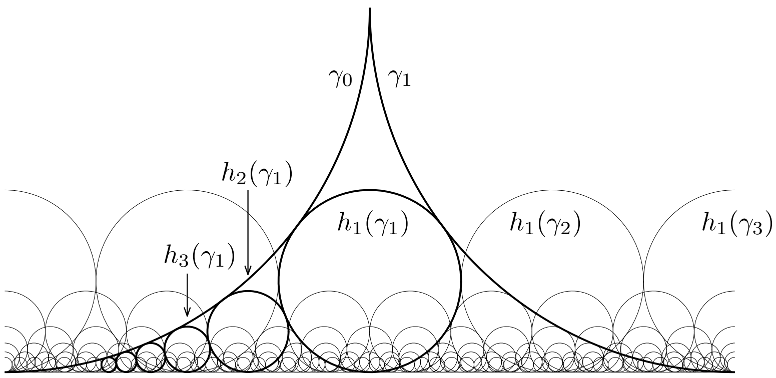

Figure 17: h_i

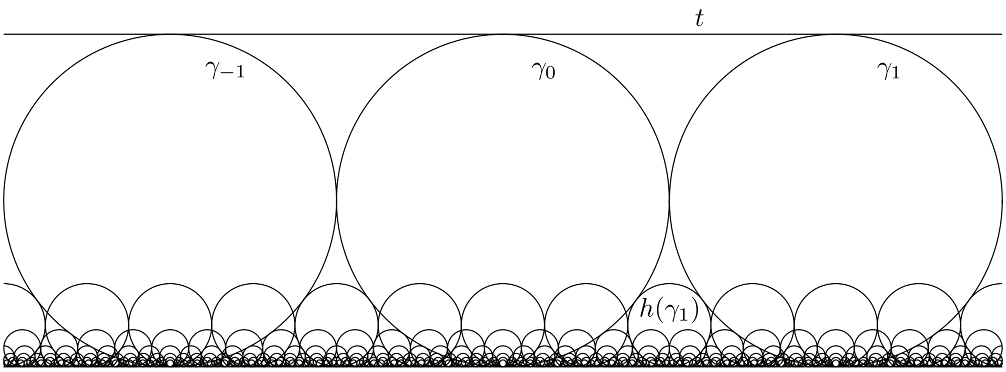

Figure 18: $\mathcal{C}$

Let $\cdots$, γ_{-2}, γ_{-1}, γ_0, γ_1, γ_2, $\cdots$ be congruent circles such that γ_i, γ_{i+1}, $\cdots$, γ_{i+k-1} form congruent circles on a line s for any integers i and $k \geq 2$. The configuration consisting of the circles and s is denoted by $\mathcal{C}_\infty$. Let h_0 be the identity mapping. Let h_1 be the homothety such that $h_1(\gamma_1) = I(\gamma_0, \gamma_1, s)$ and $h_1(s) = s$ (see Figure 17). If a homothety h_k is defined, h_{k+1} is the homothety such that $h_{k+1}(\gamma_1) = I(\gamma_0, h_k(\gamma_1), s)$ and $h_{k+1}(s) = s$. Now the homotheties h_1, h_2, $\cdots$ are defined. Let t be the remaining external common tangent of γ_0 and γ_1. Figure 18 shows the configuration $\mathcal{C} = \{t\} \cup \mathcal{C}_\infty \cup h_1(\mathcal{C}_\infty) \cup h_2(\mathcal{C}_\infty) \cdots$. Obviously the circles contained in $T(\gamma_i, \gamma_{i+1}, s)$ in $\mathcal{C}$ form $\mathcal{B}$ with sides γ_i and γ_{i+1} baseline s. By Theorems 3.2 and 5.1(ii) we get the next theorem.

Theorem 5.3. *The following circles are contained in $h_k(\mathcal{C}_\infty)$ for any integer i for the configuration $\mathcal{C}$.*
(i) *The sides of $\mathcal{A}(2k+3)$ with center circle γ_i baseline s for $k \geq 0$.*
(ii) *The inner sides of $\mathcal{B}(k^2)$ with sides γ_i and γ_{i+1} baseline s for $k \geq 1$.*

Acknowledgments. Author expresses his thanks to Professor Miwako Tonami for her useful information about Wasan books.

References

[1] Aida (会田安明) ed., Sampō Tenshōhō Shinan (算法天生法指南), 1810, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100002303
[2] Baba (馬場正統) ed., Shinseki Santei Kōhen Kaigi (真積算梯後編解義), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100003008
[3] H. Fukagawa and D. Pedoe, Japanese Temple Geometry Problems, The Charles Babbage Research Centre,Winnipeg Canada, 1989.
[4] Gunmaken Wasan Kenkyūkai (群馬県和算研究会) ed., The Sangaku in Gunma (群馬の算額), Gunmaken Wasan Kenkyūkai (群馬県和算研究会), 1987.
[5] Harada (原田保孝), Zatsudai Kaigi (雑題解義), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100004794
[6] Kawada (川田保知) et al. ed., Zoku Kiōshū (続淇澳集), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100004195
[7] Kinki Sūgakushigakukai (近畿数学史学会) ed., The Sangaku in Kinki (近畿の算額), Osaka Kyōiku Tosho (大阪教育図書), 1992
[8] Kyoto University Library (京都大学附属図書館) ed., The era of Wasan (和算の時代), Kyoto University Library (京都大学附属図書館), 2003. https://repository.kulib.kyoto-u.ac.jp/dspace/handle/2433/141905
[9] Matsunaga (松永貞辰), Sekiryū Hiritsu Endan (関流秘率演段), 1800, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100001681
[10] Matsuoka (松岡竜一), Sampō Kinuburui (算法絹篩), 1819, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005219
[11] Matsuoka (松岡 (良助) 能一), Zōho Sangaku Keiko Daizen (増補算学稽古大全), 1806, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100004852
[12] Nagai (長井忠三郎) ed., Meiji Sampō Shinsho (明治算法新書), 1881, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100018059
[13] H. Okumura, Variations of the ratio 1 : 4, Math. & Informatics Quarterly, **3** (1993) 162-166.
[14] Saitama Prefectural Library (埼玉県立図書館) ed., The Sangaku in Saitama (埼玉の算額), Saitama Prefectural Library (埼玉県立図書館), 1969.
[15] Sakabe (坂部広胖), Sampō Tenzan Shinanroku (算法点竄指南録) 1810, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100007448
[16] Shimura (志村昌義) et al. ed., Kiōshū (淇澳集), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100004194
[17] Toda (戸田広胖), Enrui Gojūmon Tōjutsu (円類五十問答術), copied date 1849, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100002467
[18] Uchida (内田秀富) ed., Zōho Sanyō Tebikigusa (増補算用手引草), 1764, Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005700
[19] Enrui Tekitōshū (円類適等集), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100003918
[20] Sanseki Mokuroku (算籍目録), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100000128
[21] Sansoku (算則), Tohoku Univ. WDB, http://www.i-repository.net/il/meta_pub/G0000398wasan_4100005018

[22] Wasan Henshū (和算編集), Tohoku Univ. WDB,
http://www.i-repository.net/il/meta_pub/G0000398wasan_4100007321

Tohoku Univ. WDB is short for Tohoku University Wasan Materials Database.