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Solution to the problem proposed in "Solution to 2017-3 Problem 5"

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Abstract. The problem proposed in "Solution to 2017-3 Problem 5" is solved and characterizations of the 3-4-5 triangle are given.

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1. Introduction

For a triangle EFG, let γ_1, γ_2, $\cdots$, γ_n be circles of radius r such that they touch the side EF from the inside of EFG, γ_1 and γ_2 touch, γ_i $(i = 3, 4, \cdots, n)$ touches γ_{i-1} from the side opposite to γ_1, γ_1 touches the side GE, γ_n touches the sides FG. In this case we say that EFG *has n circles of radius r on EF*. Those are a variety of circles called congruent circles on a line [3]. We consider the following configuration involving congruent circles on a line: Let ABC be a right triangle with hypotenuse CA and circumcircle γ with center O. Assume that a circle of radius r touches the side BC and the minor arc BC of γ at each of the midpoints, OAB has n circles of radius r on AB. This figure is denoted by $\mathcal{S}(n)$. In this case if OBC has also m circles of radius r on BC, we denote the figure by $\mathcal{S}(n, m)$. It is shown that $\mathcal{S}(2, 2)$ and $\mathcal{S}(6, 5)$ exist (see Figures 1 and 2), and the problem to determine all the existing $\mathcal{S}(n, m)$ is proposed in [1]. In this note we give a solution to this problem.

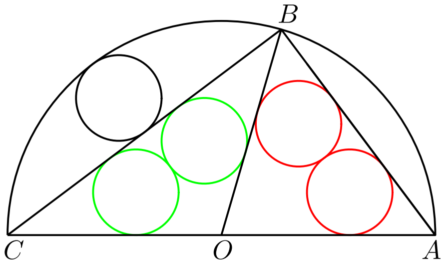

Figure 1: $\mathcal{S}(2, 2)$

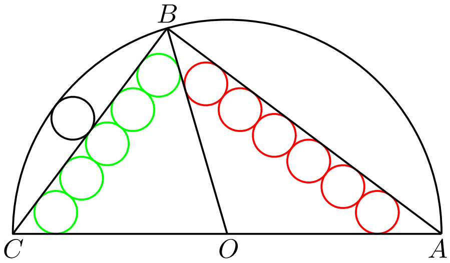

Figure 2: $\mathcal{S}(6, 5)$

2. Result

Notice that EFG has n circles of radius r on EF if and only if

$$|EF| = 2(n-1)r + r\cot(\angle GEF/2) + r\cot(\angle GFE/2).$$

Let $t = \cot(\angle ABO/2)$ for $\mathcal{S}(n)$. We use the following relation, which shows that $\mathcal{S}(n)$ is determined uniquely by n [1]:

$$t = \frac{1}{2}(\sqrt{4n+1}+1). \quad (1)$$

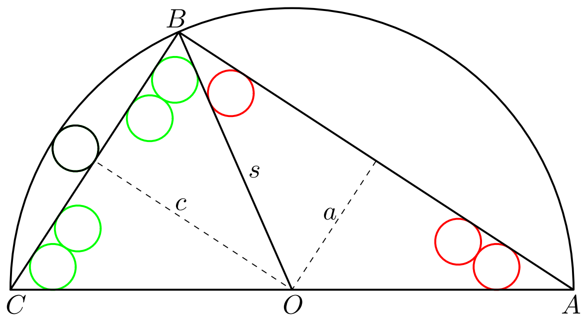

Figure 3: $\mathcal{S}(n, m)$

Theorem 1. *The configuration $\mathcal{S}(n, m)$ exists if and only if $(n, m) = (2, 2), (6, 5)$.*

Proof. We assume that $\mathcal{S}(n, m)$ exists for some integers n and m. It is sufficient to show $(n, m) = (2, 2), (6, 5)$. Let $a = |BC|/2$, $c = |AB|/2$, $s = |BO|$, $t' = \cot(\angle BCO/2)$ (see Figure 3). Obviously we have

$$s = c + 2r. \quad (2)$$

From $\tan(\angle ABO) = a/c$, we get $t = \left(c + \sqrt{a^2+c^2}\right)/a = (c+s)/a$, i.e.,

$$at = c + s. \quad (3)$$

The power of the midpoint of BC with respect to the circle γ equals

$$2r(c+s) = a^2. \quad (4)$$

Since OBC has m circles of radius r on BC, we have

$$a = (m-1)r + rt'. \quad (5)$$

Since $\angle ABO/2 + \angle BCO/2 = 45°$, we have

$$(t-1)(t'-1) = 2. \quad (6)$$

Then eliminating a, c, s, t' from (2), (3), (4), (5), (6), we get

$$2t^2 - (m+2)t + m - 2 = 0. \quad (7)$$

Substituting (1) in (7), and solving the resulting equation for m, we get

$$m = \frac{1}{n}(n-1)(\sqrt{4n+1}+1). \quad (8)$$

Since n and $n-1$ are related to prime numbers, (8) shows that $\sqrt{4n+1}+1$ is a multiple of n, i.e., $\sqrt{4n+1}$ is an integer. Therefore $4n+1 = (2k+1)^2$ for a positive integer k. Then we get $n = k(k+1)$ and

$$m = 2k + 2 - \frac{2}{k}$$

by (8). This implies that $2/k$ is an integer. Therefore $k = 1, 2$, i.e., $(n, m) = (2, 2), (6, 5)$. The proof is now complete. □

3. CHARACTERIZATIONS OF 3-4-5 TRIANGLES

As noted in [1], $\mathcal{S}(2)$ and $\mathcal{S}(6)$ are only the pair which can be derived from the same triangle, where the triangle is a 3-4-5 triangle (see Figure 4). Hence the right triangles in $\mathcal{S}(2, 2)$ and $\mathcal{S}(6, 5)$ are also 3-4-5 triangles.

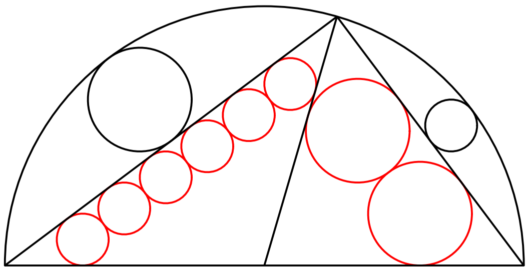

Figure 4: $\mathcal{S}(2)$ and $\mathcal{S}(6)$

Therefore we get a characterization of the 3-4-5 triangle by Theorem 1: *a right triangle is a 3-4-5 triangle if and only if it satisfies one of the followings:*
(i) $\mathcal{S}(n)$ *and* $\mathcal{S}(m)$ *are derived from the triangle for some distinct integers* n *and* m.
(ii) $\mathcal{S}(2)$ *or* $\mathcal{S}(6)$ *is derived from the triangle.*
(iii) $\mathcal{S}(n, m)$ *is derived from the triangle for some integers* n *and* m.
(iv) $\mathcal{S}(2, 2)$ *or* $\mathcal{S}(6, 5)$ *is derived from the triangle.*

For another configuration involving congruent circles on a line and 3-4-5 triangles see [2].

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