Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 3 (2019), pp. 123-125
Received 20 November 2019. Published on-line 23 November 2019
web: http://www.sangaku-journal.eu/

Remarks on a problem involving four circles in a parallelogram

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We generalize the problem in [2] involving four circles in a parallelogram.

Keywords. parallelogram, isosceles trapezoid, golden number.

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

We consider a problem in Wasan geometry involving a parallelogram, which was rarely considered in Wasan. Let $ABCD$ be a quadrilateral such that the sides DA and BC are parallel (see Figure 1). We assume that α, β, γ, δ, ε are circles of radii a, b, c, d, e, respectively, lying inside of $ABCD$ such that ε touches BC, CD and DA; γ is the incircle of the curvilinear triangle made by BC, CD and ε; δ is the incircle of the curvilinear triangle made by CD, DA and ε; α touches DA, AB and ε externally; β touches AB, BC and α, ε externally. We denote the configuration consisting of $ABCD$ and the five circles by $\mathcal{Q}$. If $ABCD$ is a parallelogram for $\mathcal{Q}$, we say that $\mathcal{Q}$ has a parallelogram. We generalize the following problem in [2] (see Figure 2).

Problem 1. $\mathcal{Q}$ has a parallelogram and $a = d$. Find the value e/a.

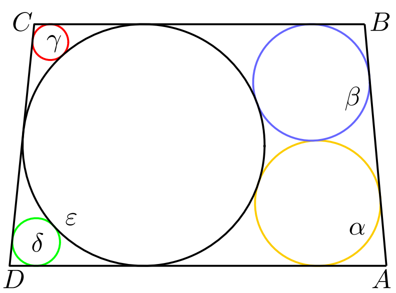

Figure 1: The configuration $\mathcal{Q}$.

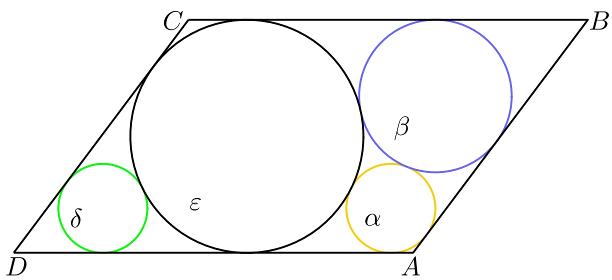

Figure 2: $\mathcal{Q}$ with $AB \parallel CD$, $a = d$.

2. Generalization

We give a condition in which $\mathcal{Q}$ has a parallelogram in a general case $a \neq d$. We use the next theorem [1]. A proof of this can be found in [4].

Theorem 1. *The following relation holds for* $\mathcal{Q}$.

$$a = e^2/(4b). \tag{1}$$

Theorem 2. *The configuration* $\mathcal{Q}$ *has a parallelogram if and only if*

$$\sqrt{b} = \frac{\sqrt{d} + \sqrt{e}}{2}. \tag{2}$$

Proof. Assume that $\mathcal{Q}$ has a parallelogram. Let $t = \tan(D/2)$. Then we have

$$t = \tan\frac{B}{2} = \frac{e-d}{2\sqrt{de}}, \tag{3}$$

and $\tan(A/2) = \tan(C/2) = 1/t$ (see Figure 3). From $|DA| = |BC|$ we have

$$d/t + 2\sqrt{de} + 2\sqrt{ea} + at = b/t + 2\sqrt{be} + et.$$

Substituting (1) and (3) in the last equation and rearranging, we get

$$\frac{(2\sqrt{b} - \sqrt{d} - \sqrt{e})(2\sqrt{b} - \sqrt{d} + \sqrt{e})(2\sqrt{bd} + e - \sqrt{de})(2\sqrt{bd} + e + \sqrt{de})}{8b(d-e)\sqrt{d}} = 0.$$

Therefore we get (2). Conversely, we assume (2). Let d' be the inradius of the curvilinear triangle made by ε, DA and the tangent of ε which forms a parallelogram with the lines DA, AB and BC containing ε. Then we get $\sqrt{b} = (\sqrt{d'} + \sqrt{e})/2$ as we have just proved, i.e., $d = d'$. Hence the tangent coincides with CD. Therefore $ABCD$ is a parallelogram. □

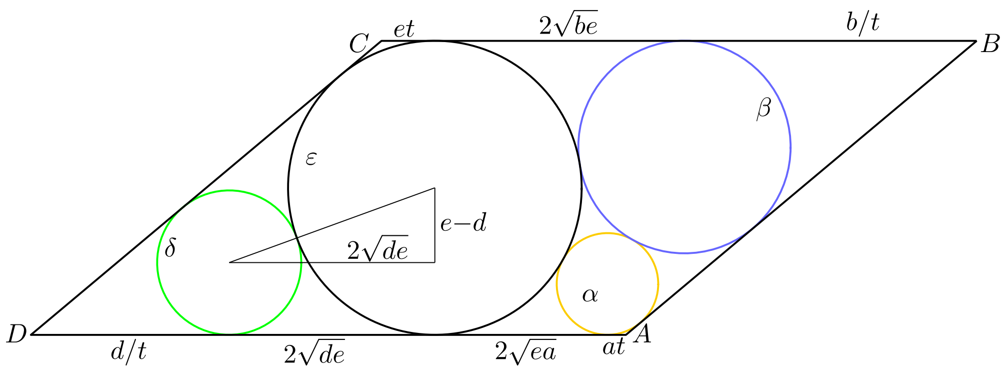

Figure 3.

The next corollary involving two congruent circles and the golden number gives an answer of the problem. A theorem involving two congruent circles and the golden number can be found in [3].

Corollary 1. *If* $\mathcal{Q}$ *has a parallelogram, then the circles* α *and* δ *are congruent if and only if*

$$\sqrt{\frac{e}{a}} = \frac{1+\sqrt{5}}{2}.$$

Proof. Eliminating b from (1) and (2), we get $e = \sqrt{a}(\sqrt{d} + \sqrt{e})$, which is equivalent to

$$\left(\sqrt{\frac{e}{a}} - \frac{1+\sqrt{5}}{2}\right)\left(\sqrt{\frac{e}{a}} - \frac{1-\sqrt{5}}{2}\right) = \sqrt{\frac{d}{a}} - 1.$$

□

Corollary 2. *If $\mathcal{Q}$ has a parallelogram, then $2\sqrt{b} + \sqrt{c} = 2\sqrt{a} + \sqrt{d}$.*

Proof. The corollary follows from (2) and $\sqrt{a} = (\sqrt{c} + \sqrt{e})/2$. □

3. Isosceles trapezoid

We consider the case in which $ABCD$ is an isosceles trapezoid with $|AB| = |CD|$ for $\mathcal{Q}$ (see Figure 4). In this case we say that $\mathcal{Q}$ has an isosceles trapezoid. Let φ be the reflection in the line parallel to BC passing through the center of ε. If $\mathcal{Q}$ has an isosceles trapezoid, then we can get another $\mathcal{Q}$ having a parallelogram by replacing α, β and AB by $\varphi(\beta)$, $\varphi(\alpha)$ and $\varphi(AB)$, respectively with appropriate relabeling, and vise versa (see Figure 5). Therefore we get the next theorem.

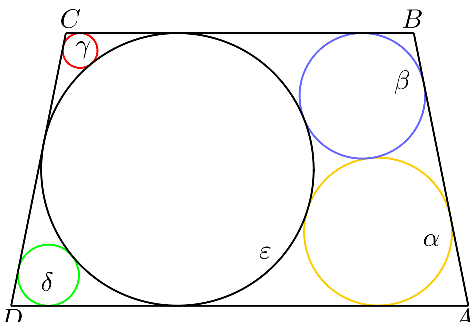

Figure 4.

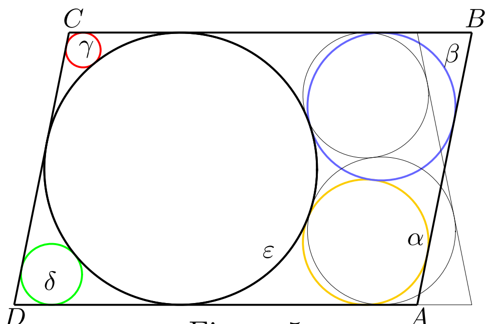

Figure 5.

Theorem 3. *The configuration $\mathcal{Q}$ has an isosceles trapezoid if and only if*

$$\sqrt{a} = \frac{\sqrt{d} + \sqrt{e}}{2}.$$

In this event, the following statements hold.
(i) *The circles β and δ are congruent if and only if*

$$\sqrt{\frac{e}{b}} = \frac{1+\sqrt{5}}{2}.$$

(ii) *The relation $2\sqrt{a} + \sqrt{c} = 2\sqrt{b} + \sqrt{d}$ holds.*

References

[1] Aida (会田安明) ed., Sampō Tenshōhō (算法天生法) 1788, Tohoku University Digital Collection.
[2] Kimura (木村尚寿), Onchi Sansō (温知算叢), 1828, Tohoku University Digital Collection.
[3] H. Okumura, A characterization of the golden arbelos involving an Archimedean circle, Sangaku J. Math., **3** (2019) 67–71.
[4] H. Okumura, Configurations of congruent circles on a line, Sangaku J. Math., (2017) 24–34.