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On a Sangaku Problem Involving the Incircles in a Square Tiling: Solutions to Problems 2019-2

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Abstract. We solve and generalize a sangaku problem involving the incircles in a square tiling, which appears in [1] and [2]. Our proofs are all purely geometrical.

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1. Introduction

The following problem originates from the sangaku in Yamagata hung in 1881, which was proposed by Gotoh [1]. It was pointed out by H. Okumura in this journal, who asked for a proof and a generalization [2]. In this note we present a geometrical approach to the problem together with some interesting extensions and related results.

It will be convenient for our approach to introduce the points in this configuration in a slightly different order than in [2].

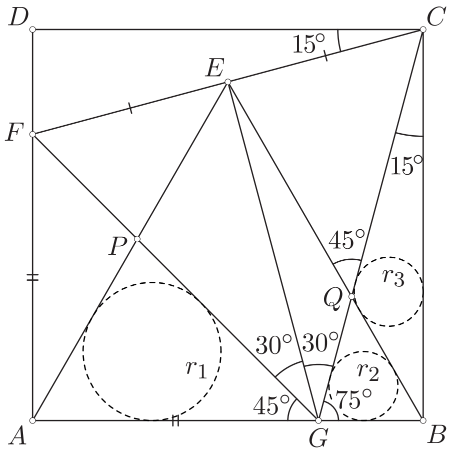

Figure 1.

Namely, let $ABCD$ be a square (see Figure 1). Point E is chosen inside the square in such a way that triangle ABE is equilateral. Then since $BC = BE$ and $\angle CBE = 30°$, we infer that $\angle BCE = \angle BEC = 75°$, and hence $\angle ECD = 15°$.

Furthermore, denote by F the intersection point of lines CE and DA. Since E lies on the perpendicular bisector of AB, which is parallel to both lines AF and BC, we get $EC = EF$. Finally, let G be the reflection of point F with respect to diagonal AC. Then G lies on AB, $CF = CG$, and $\angle FCG = 90° - 2 \cdot 15° = 60°$. Therefore, triangle CFG is equilateral.

Set $P = AE \cap FG$ and $Q = BE \cap CG$. Moreover, let r_1, r_2, and r_3 be the inradii of triangles AGP, BGQ, and BCQ, respectively. The sangaku problem presented in [1] and [2] reads as follows.

Theorem 1.1. *The following formulas hold:* $r_1 = 2r_2$ *and* $r_2 = r_3$.

In the next section we present two general theorems, which immediately imply both equalities of Theorem 1.1.

2. Two general theorems

Before we present the first theorem, we make some additional observations at Figure 1. Since $AF = AG$, we have $\angle AGP = 45°$. Point E is the midpoint of side CF of equilateral triangle CFG, which gives $\angle PGE = \angle QGE = 30°$. Since points F and G are symmetric in AC, $\angle BGQ = 90° - 15° = 75°$.

Theorem 2.1. *Let ABE be an equilateral triangle and let G be any point lying on side AB (see Figure 2). Points P and Q lie on sides AE and BE, respectively, and satisfy $\angle PGE = \angle QGE = 30°$. Set $\alpha = \angle AGP$ and $\beta = \angle BGQ$. Denote by r_1 and r_2 the inradii of triangles AGP and BGQ, respectively. Then*

$$\frac{r_1}{r_2} = \frac{\sin 2\alpha}{\sin 2\beta}\,.$$

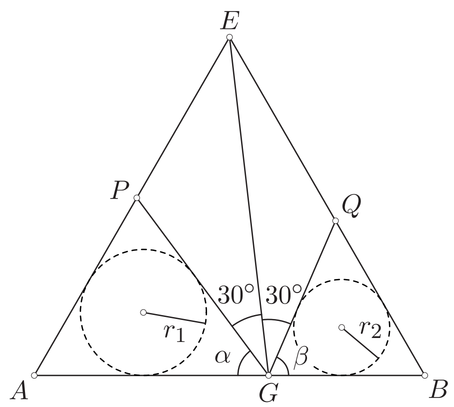

Figure 2.

Remark. Setting $\alpha = 45°$ and $\beta = 75°$ in the above formula, we obtain $r_1 = 2r_2$. This proves the first equality of Theorem 1.1.

Proof. Observe that $\alpha + \beta + 60° = 180°$. Since the angles of triangle AGP at vertices A and G have measures $60°$ and α, respectively, the last equality implies that $\angle APG = \beta$ (see Figure 3). Similarly, $\angle BQG = \alpha$. Therefore, triangles AGP

and BQG are similar, so

$$\frac{r_1}{r_2} = \frac{PG}{QG}. \tag{1}$$

Let X and Y be the feet of the perpendiculars from E onto lines PG and QG, respectively. Since E lies on the angle-bisector of $\angle PGQ$, we have $EX = EY$. Moreover, $\angle XEY = 180° - \angle XGY = 120°$. Thus

$$\angle XEP + \angle YEQ = 120° - \angle PEQ = 60° = \angle PEQ. \tag{2}$$

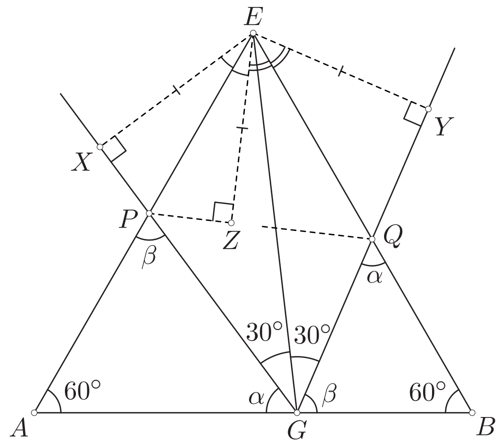

Figure 3.

Denote by Z the reflection of X with respect to line AE. Then $EZ = EX = EY$ and from equality (2) we obtain

$$\angle YEQ = \angle PEQ - \angle XEP = \angle PEQ - \angle PEZ = \angle ZEQ.$$

This implies that triangles YEQ and ZEQ are congruent. Hence we infer that $\angle EZQ = 90° = \angle EZP$. Therefore point Z lies on segment PQ, which gives $\angle QPE = \angle ZPE = \angle XPE = \beta$. Analogously, $\angle PQE = \alpha$. Consequently, applying the sine law for triangle PQG, we get

$$\frac{PG}{QG} = \frac{\sin(180° - 2\alpha)}{\sin(180° - 2\beta)} = \frac{\sin 2\alpha}{\sin 2\beta}. \tag{3}$$

Combining equalities (1) and (3), we obtain the desired formula. □

Before we proceed with the next theorem, we come back again to Figure 1. Concentrate this time on triangle BCG and recall that $\angle CGB = 75°$, $\angle GCB = 15°$. This yields $\angle BQG = 180° - 60° - 75° = 45° = \frac{1}{2}(75° + 15°)$.

Theorem 2.2. *Let BCG be a triangle with $\angle CGB = \alpha$, $\angle GCB = \beta$, and $\alpha > \beta$ (see Figure 4). Point Q lies on segment CG and satisfies $\angle BQG = \frac{1}{2}(\alpha + \beta)$. Denote by r_2 and r_3 the inradii of triangles BGQ and BCQ, respectively. If*

$$\frac{\sin(\alpha + \beta)}{2} = \sin\Big(\frac{\alpha - \beta}{2}\Big), \tag{4}$$

then $r_2 = r_3$.

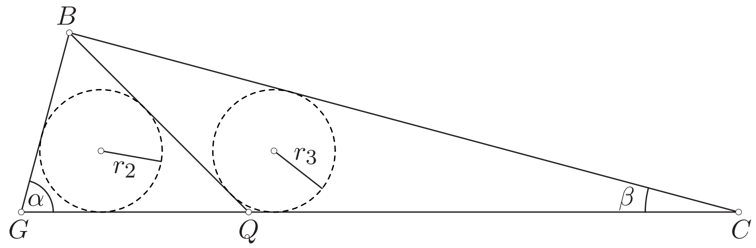

Figure 4.

Remark. Note that $\alpha = 75°$ and $\beta = 15°$ satisfy condition (4). Therefore, Theorem 2.2 implies the second equality of Theorem 1.1.

Proof. Denote by ω the incircle of triangle BCQ (see Figure 5). Let X and Y be points lying on lines BC and BQ, respectively, such that lines XY and BG are parallel and ω is the incircle of triangle BXY. Observe that

$$\angle BXY = 180° - \angle GBC = \alpha + \beta \quad \text{and} \quad \angle XBY = \angle BQG - \angle BCG = \frac{1}{2}(\alpha - \beta)\,.$$

Using (4) and the sine law for triangle BXY, we obtain $BY = 2XY$.

Let M be the midpoint of segment BY and let I be the center of circle ω. Then $XY = MY$, which implies that triangles XIY and MIY are congruent. Therefore, we obtain

$$\angle IMY = \angle IXY = \frac{1}{2}\angle BXY = \frac{1}{2}(\alpha + \beta) = \angle BQG\,.$$

Hence lines IM and CG are parallel.

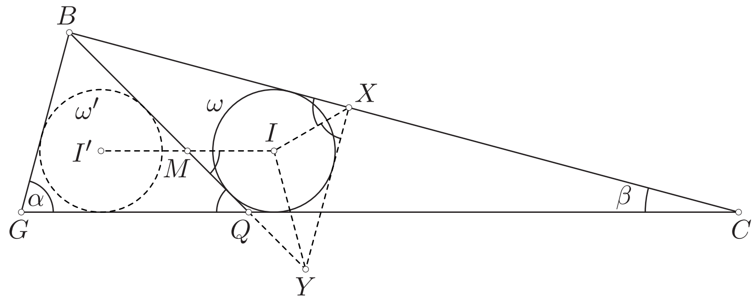

Figure 5.

Let ω' be the image of circle ω under the symmetry with center M. Denote also by I' the center of circle ω'. Since IM and CG are parallel, lines II' and CG are also parallel. Moreover, circles ω and ω' are congruent and ω is tangent to CG, so ω' is tangent to CG, as well.

The symmetry takes line XY to line BG, so ω' is tangent to BG. Furthermore, since BY is taken by the symmetry to itself, circle ω' is also tangent to BY. This implies that ω' is the incircle of triangle BGQ, which implies $r_2 = r_3$. □

References

[1] A. Hirayama, M. Matsuoka ed., The sangaku in Yamagata, 1966.

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