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Arbeloi determined by a chord and solutions to Problems 2017-3-8 and 2019-3-4

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Abstract. A chord of a circle δ with two circles touching δ internally and the chord at the midpoint determines two congruent arbeloi. In this paper we consider the radius of Archimedean circles of such arbeloi determined by a circle and its chords. There are several theorems in Wasan geometry involving Archimedean circles of several arbeloi determined by a circle and its chords. We give simple proofs of those theorems. Solutions of Problems 2017-3-8 and 2019-3-4 are also given.

Keywords. arbeloi determined by a chord

Mathematics Subject Classification (2010). 01A27, 51M04, 51N20.

1. Introduction

We consider an arbelos formed by three semicircles α, β and γ of diameters AO, BO and AB, respectively for a point O on the segment AB. We denote the arbelos by (α, β, γ) and call the radical axis of α and β the axis of (α, β, γ). Let a and b be the radii of α and β, respectively. Circles of radius $r_{\mathrm{A}} = ab/(a+b)$ are called Archimedean circles of (α, β, γ) or are said to be Archimedean with respect to (α, β, γ). The circle touching α (resp. β) externally γ internally and the axis from the side opposite to B (resp. A) is Archimedean with respect to (α, β, γ).

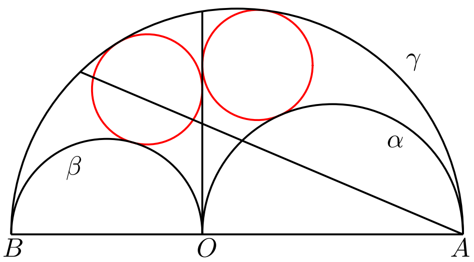

Figure 1: (α, β, γ)

In the first half of this paper, we give a solution of Problem 2017-3-8 [16] and consider a chord of γ passing through the point A as shown in Figure 1. We show that the chord yields several Archimedean circles of (α, β, γ) in a certain case.

Let $\hat{\gamma}$ be the circle made by the semicircle γ and its reflection in AB. A chord of $\hat{\gamma}$ passing through A with two circles touching $\hat{\gamma}$ internally and the chord at the midpoint forms two congruent arbeloi such that the chord is the axis of the arbeloi. We show that the radius of Archimedean circles of the arbeloi can be expressed in a simple way.

In the last part we consider the radius of Archimedean circles of the arbeloi determined by a given circle and its chord. There are several theorems in Wasan geometry stating relationships between the radius of the circle and the radii of Archimedean circles of several arbeloi determined by the circle and its chords in the case in which the chords form a triangle or a quadrilateral. We give simple proofs of those theorems and a solution of Problem 2019-3-4 [15].

2. Solution of Problem 2017-3-8

Let J be the point of intersection of γ and the perpendicular bisector of AO, and let δ be the circle touching the segment AJ at the midpoint and the minor arc AJ of γ internally for (α, β, γ). Problem 2017-3-8 may be stated as follows (see Figure 2):

Problem 1. If the semicircle β and the circle δ have the same radius, show $a = 7b$.

In this section we give several characterizations of the figure in the problem together with a solution. Let I be the point of intersection of γ and the axis. Considering the power of the point O with respect to γ, we get $|IO| = 2\sqrt{ab}$. We use a rectangular coordinate system with origin O such that the farthest point on α has coordinates (a, a). For a point P, P_f denotes the foot of perpendicular from P to AB. We denote the distance between the center of a circle ζ and a line l by $d_\zeta(l)$. The next theorem gives characterizations of the figure in the problem and a solution of the problem.

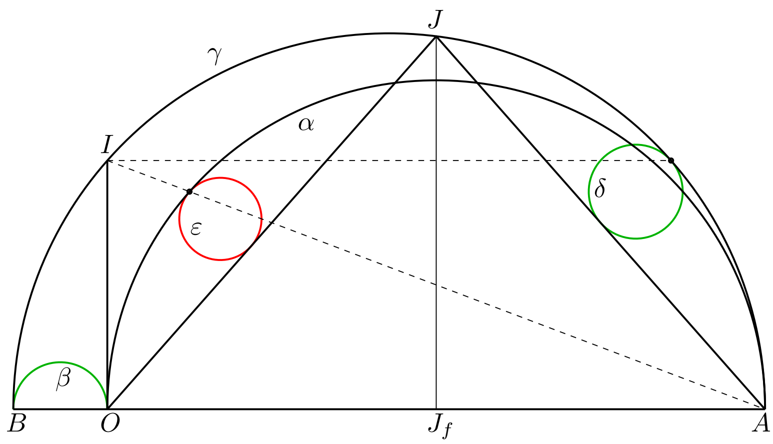

Figure 2.

Theorem 1. *If ε is the circle touching the segment JO and the minor arc of α cut by JO internally at the midpoint, then the following statements are equivalent.*
(i) *The semicircle β and the circle δ have the same radius.*
(ii) *The relation $a = 7b$ holds.*

(iii) *The circle* ε *is Archimedean with respect to* (α, β, γ).
(iv) *The point of tangency of* γ *and* δ *is the reflection of* I *in the line* JJ_f.
(v) *The point of tangency of* α *and* ε *lies on the segment* AI.

Proof. Let $m = \sqrt{a+2b}$. Then the point J has coordinates $(a, m\sqrt{a})$ and the line AJ has an equation $mx + \sqrt{a}y - 2am = 0$. Since the center of γ has coordinates $(a-b, 0)$, we have

$$d_\gamma(AJ) = \frac{|m(a-b) - 2am|}{\sqrt{2(a+b)}} = m\sqrt{\frac{a+b}{2}}.$$

While the diameter of δ equals $a + b - d_\gamma(AJ)$. Therefore (i) is equivalent to $2b = a + b - d_\gamma(AJ)$, which is equivalent to (ii), i.e. (i) and (ii) are equivalent.

Since $mx - \sqrt{a}y = 0$ is an equation of the line JO, $d_\alpha(JO) = ma/\sqrt{2(a+b)}$. While the diameter of ε equals $a - d_\alpha(JO)$. Therefore (iii) is equivalent to $a - d_\alpha(JO) = 2r_{\mathrm{A}}$, which is also equivalent to (ii), i.e. (ii) and (iii) are equivalent.

The point of tangency of γ and δ coincides with the point of intersection of γ and the perpendicular bisector of AJ, and has y-coordinate $\sqrt{a(a+b)/2}$. While I has y-coordinate $2\sqrt{ab}$. Then $\sqrt{a(a+b)/2} = 2\sqrt{ab}$ is equivalent to (ii), i.e., (ii) and (iv) are equivalent.

The point of tangency of α and ε coincides with the point of intersection of α and the perpendicular from J_f to JO and has x-coordinate $a - am/\sqrt{2(a+b)}$. Since the point of intersection of α and AI has x-coordinate $2r_{\mathrm{A}}$, the two x-coordinates coincide if and only if (ii) holds, (ii) and (v) are equivalent. □

3. THE REFLECTION OF THE LINE JO IN THE AXIS

In this section we consider another point H on γ, and consider the case in which the line HO is the reflection of the line JO in the axis. Several new Archimedean circles of (α, β, γ) are obtained in this case.

3.1. **Fundamental properties.** For two points P and Q, we denote the circle of center P passing through Q by $P(Q)$.

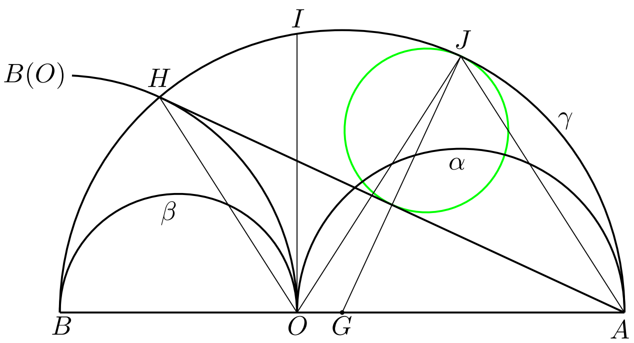

Figure 3.

Theorem 2. *The following statement are equivalent for a point* H *on* γ.
(i) *The circle touching* AH *at the midpoint and the minor arc* AH *of* γ *internally has radius* $a/2$.
(ii) *The point* H *coincides with the point of intersection of* γ *and the circle* $B(O)$.

(iii) *The midpoint of the minor arc AH of γ coincides with J.*
(iv) *The reflection of the line JO in the axis coincides with the line HO.*

Proof. Let G be the center of γ (see Figure 3). The part (i) is equivalent to $d_\gamma(AH) = b$, or the line AH has an equation

(1) $$b(x - 2a) + m\sqrt{a}y = 0,$$

since G has coordinates $(a - b, 0)$, where recall $m = \sqrt{a + 2b}$. This is equivalent to that H has coordinates

(2) $$(-2r_{\rm A}, 2r_{\rm A}m/\sqrt{a}),$$

which coincides with the point of intersection of γ and $B(O)$. Hence (i) and (ii) are equivalent. The part (iii) is equivalent to that the lines AH and GJ are perpendicular, while the line GJ has an equation

$$m\sqrt{a}(x - (a - b)) - by = 0.$$

Hence (i) and (iii) are equivalent. The equivalence of (ii) and (iv) follows from that the slope of OJ equals $m/\sqrt{a}$ and the slope of the line joining O and the point of coordinates (2) equals $-m/\sqrt{a}$. □

3.2. **A rhombus with Archimedean incircle.** We show that there are several new Archimedean circles of (α, β, γ) if HO is the reflection of JO in the axis, where the words "with respect to (α, β, γ)" will be omitted. We now assume Theorem 2(i) and that M is the remaining point of intersection of α and AH, N is the point of intersection of β and BH, α meets JO and JA in points E and F, respectively, and the line BJ meets AH, HO and NO in points S, T and U, respectively (see Figure 4).

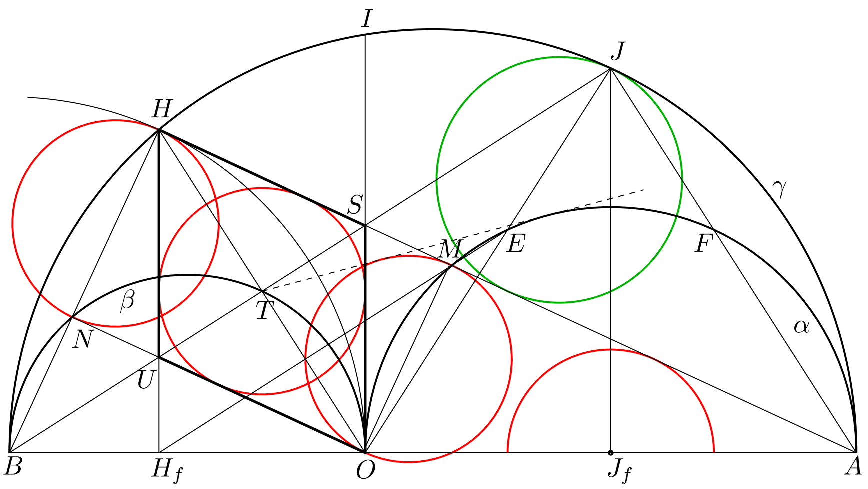

Figure 4.

We denote the circle of diameter PQ by (PQ) for two points P and Q. The line BJ has an equation

(3) $$-\sqrt{a}(x + 2b) + my = 0.$$

Hence the point S has coordinates $(0, 2b\sqrt{a}/m)$ by (1), i.e., S lies on the axis. While the line NO is parallel to AH and has equation $bx + m\sqrt{a}y = 0$ by (1). This implies that U has x-coordinate $-2r_{\rm A}$ and lies on HH_f by (2). Since the circle (MO) is Archimedean [10], and $MHNO$ is a rectangle, the circle (HN) is

Archimedean, while the distance between the axis and the line HH_f equals $2r_{\rm A}$. Therefore $SHUO$ is a rhombus with Archimedean incircle of center T, where T has coordinates $(-r_{\rm A}, r_{\rm A}m/\sqrt{a})$ and lies on β. The circle of center J_f touching AH is Archimedean. The point J is the external center of similitude of α and the circle (EF), which is Archimedean [3, 9]. Therefore the fact $|TJ_f| = m\sqrt{a} = |JJ_f|$ implies that T is also the external center of similitude of α and the Archimedean circle (MO), i.e., the tangent of α from T also touches (MO). The point M has coordinates $(2r_{\rm A}^2/a, 2r_{\rm A}m\sqrt{a}/(a+b))$ and the line H_fM has equation $-\sqrt{a}(x + 2r_{\rm A}) + my = 0$, which is parallel to BJ by (3) and passes through the point E whose coordinates are $(a - r_{\rm A}, \sqrt{a^2 - r_{\rm A}^2})$.

3.3. **Schoch line and the triangle HOJ.** The perpendicular to AB expressed by the equation $x = s$, where $s = r_{\rm A}(b-a)/(a+b)$, is called Schoch line, which passes through the center of the Archimedean circle touching γ internally and the circles $B(O)$ and $A(O)$ externally [4], [10] (see Figure 5). We consider properties of the triangle HOJ related to this line. The proofs are straight forward and will be omitted.

Theorem 3. *The incircle of the triangle HOJ has center S and radius $\sqrt{2abr_{\rm A}}/m$, and touches the side HO at T and the side HJ at the point of intersection of HJ and Schoch line.*

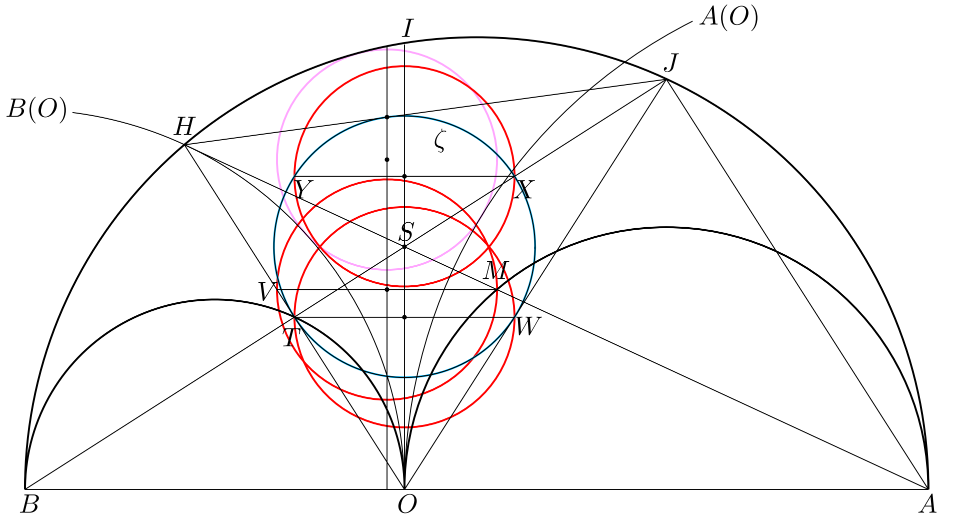

Figure 5.

Let ζ be the incircle of HOJ. The point of tangency of ζ and HJ has coordinates $(s, (3a+b)b\sqrt{a}m/(a+b)^2)$. Let W be the reflection of T in the axis. Then ζ touches JO at W and the circle (TW) is Archimedean. If V is the reflection of M in Schoch line, then V lies on HO and the circle (MV) is Archimedean. If X is the reflection of T in the point S, and Y is the reflection of X in the axis, then X lies on ζ and BJ and the circle (XY) is Archimedean. Notice that Y also lies on ζ.

4. Arbeloi determined by a circle and its chord

In this section we consider the two arbeloi formed by a circle δ and two circles touching δ internally and a chord of δ at the midpoint. We will show simple relationships between the radius of Archimedean circles of the arbeloi and the chord. The case in which the chord is a side of a triangle or a quadrilateral was considered in Wasan geometry. We consider the case in the last part.

4.1. **Arbelos determined by a chord passing through the point A.** Let $\hat{\gamma}$ be the circle made by γ and its reflection in AB. For a point H on γ, we consider the two congruent arbeloi formed by $\hat{\gamma}$ and the two circles touching $\hat{\gamma}$ internally and the chord AH at the midpoint (see Figure 6). We denote the radius of Archimedean circles of these arbeloi by $r(H)$.

Theorem 4. $r(H) = \dfrac{|AH_f|}{8}$.

Proof. Since the radii of the two inner circles forming the arbeloi are $(a + b - d_\gamma(AH))/2$ and $(a + b + d_\gamma(AH))/2$, we get

$$r(H) = \frac{(a+b)^2 - d_\gamma(AH)^2}{4(a+b)}. \tag{4}$$

Let $|AH_f| = h$. Then $|H_f H|^2 = h|BH_f| = 2(a + b)h - h^2$. Since the midpoint of AH has coordinates $(2a - h/2, |H_f H|/2)$, we get

$$\begin{aligned} d_\gamma(AH)^2 &= \left(a + b - \frac{h}{2}\right)^2 + \frac{|H_f H|^2}{4} \\ &= (a+b)^2 - (a+b)h + \frac{h^2}{4} + \frac{(a+b)h}{2} - \frac{h^2}{4} = (a+b)^2 - \frac{(a+b)h}{2}. \end{aligned}$$

Therefore by (4) we get

$$r(H) = \frac{(a+b)h}{2} \frac{1}{4(a+b)} = \frac{h}{8}.$$

□

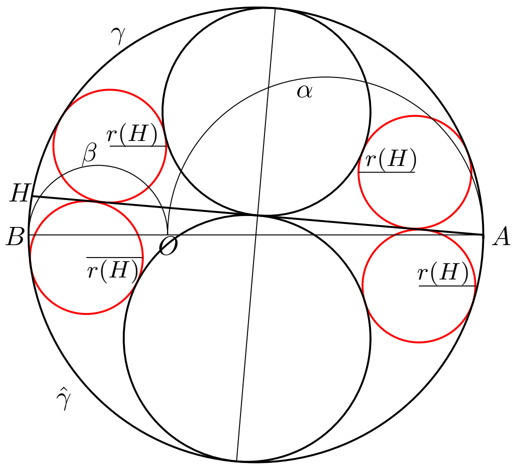

Figure 6.

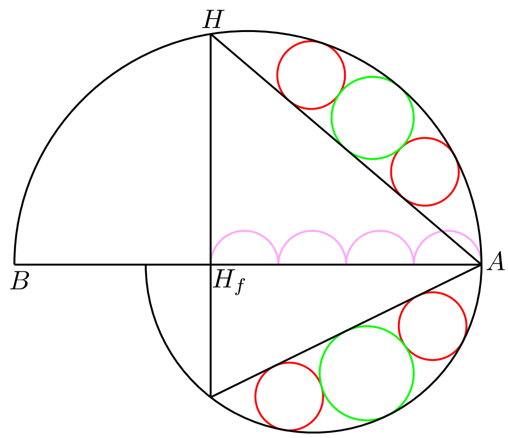

Figure 7.

Notice that $r(H)$ depends only on $|AH_f|$. Hence the four red circles in Figure 7 are congruent, while the two green circles are not. The four red circles and four pink semicircles also have the same radius. If $z < a/4$, then $8z < 2a$, i.e., there is a point H on γ such that $|AH_f| = 8z$ for any $b > 0$. Therefore there is a point H on γ such that $r(H) = z$ for a real number $z < a/4$. Let K be the farthest point on γ from AB.

Corollary 1. *The following statements are true.*
(i) $r(I) = a/4$.
(ii) $r(J) = a/8$.
(iii) $r(K) = (a + b)/8$.

A problem related to Corollary 1(ii) in the case $a = b$ can be found in a sangaku presented in 1808 in Chiba [6], which was proposed by Kimitsuka or Kimizuka (君塚喜三郎). A problem stating Corollary 1(iii) can be found in the sangaku hung in 1865 in Gifu [12], which was proposed by Shichi (志智孝成). Essentially the same problem can also be found in [13]. The next corollary follows from Corollary 1 (see Figures 8, 9, 10).

Corollary 2. *The following statements are true for* (α, β, γ).
(i) $r(I) = r_{\mathrm{A}}$ *if and only if* $a = 3b$.
(ii) $r(J) = r_{\mathrm{A}}$ *if and only if* $a = 7b$.
(iii) $r(K) = r_{\mathrm{A}}$ *if and only if* $a = \left(1 \pm \sqrt{2}\right)^2 b$.

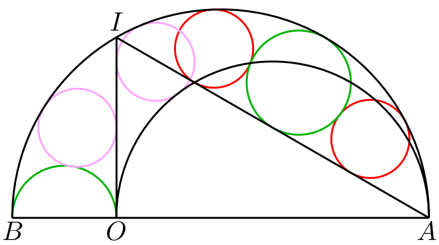

Figure 8: $a = 3b$

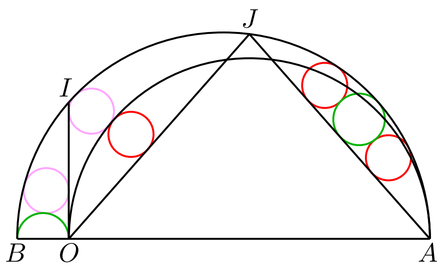

Figure 9: $a = 7b$

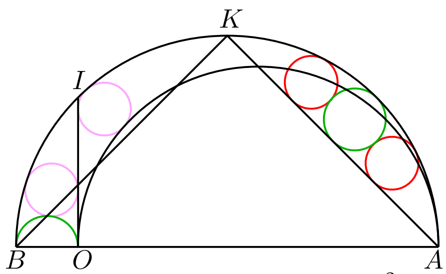

Figure 10: $a = \left(1 + \sqrt{2}\right)^2 b$

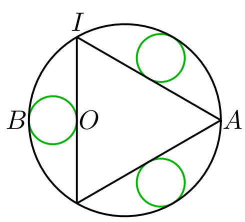

Figure 11.

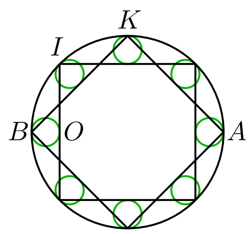

Figure 12.

Corollary 2(ii) gives another characterization of the figure of Problem 1 (see Figure 9). Figures 11 is obtained from Figure 8 with its reflection in the line AB by removing the Archimedean circles of (α, β, γ), where the triangle is equilateral. Figure 12 is obtained from Figure 10 similarly, with several additional line segments and green circles.

4.2. **Triangle case.** We now consider a triangle ABC with circumcircle δ with the standard notations $|AB| = c$, $|BC| = a$ and $|CA| = b$ (see Figure 13). If a chord of δ has length a, then $\bar{a}$ denotes the radius of Archimedean circles of the arbeloi formed by δ and two circles touching δ internally and the chord at the midpoint.

Theorem 5. *For a triangle ABC with circumradius R, we have $c = 4\sqrt{R\bar{c}}$.*

Proof. If a' and b' are the radii of the two circles touching the circumcircle internally and AB at the midpoint, then $R = a' + b'$ and $c/2 = 2\sqrt{a'b'}$. Hence we get

$$\bar{c} = \frac{a'b'}{a'+b'} = \frac{c^2}{16R}.$$

□

The only if part of the next theorem is Problems 2019-3-4 [15], which was presented by Kawada (川田保知) in a sangaku dated 1816 [11, 14] (see Figure 14).

Theorem 6. *ABC is a right triangle with right angle at C if and only if $\bar{a}+\bar{b}=\bar{c}$.*

Proof. We get $a^2+b^2-c^2 = 16R(\bar{a}+\bar{b}-\bar{c})$ by Theorem 5. □

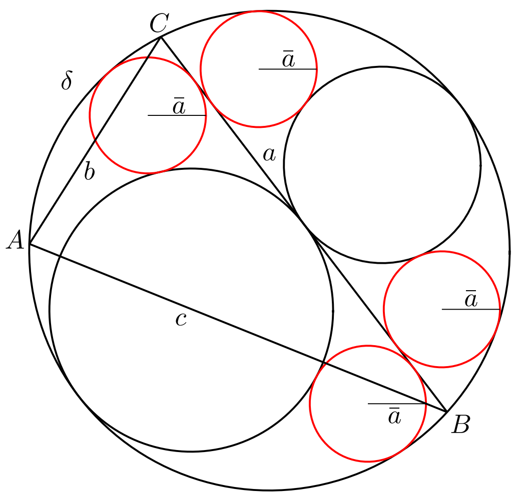

Figure 13.

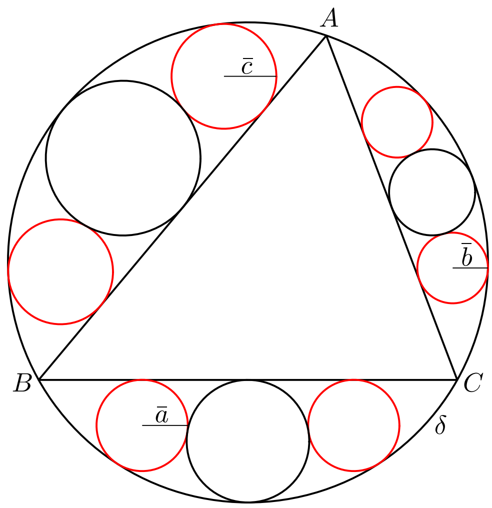

Figure 14.

The next theorem can be found in [1]. The same theorem was also given in a sangaku problem proposed by Nakamura (中村幸蔵永著) in 1796 [5].

Theorem 7. *For a triangle ABC with circumradius R, we have*

$$R = \frac{16\bar{a}\bar{b}\bar{c}}{2(\bar{a}\bar{b}+\bar{b}\bar{c}+\bar{c}\bar{a})-\bar{a}^2-\bar{b}^2-\bar{c}^2}. \tag{5}$$

Proof. We use the relation between the sides of ABC and the circumradius:

$$R = \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}. \tag{6}$$

Substituting $a = 4\sqrt{R\bar{a}}$, $b = 4\sqrt{R\bar{b}}$ and $c = 4\sqrt{R\bar{c}}$ in (6) and rearranging, we get (5). □

4.3. **Cyclic quadrilateral.** We will give proofs of three theorems in Wasan geometry. We consider a cyclic quadrilateral of side lengths a, b, c, d with circumradius R, where we assume that the two sides of lengths a and c have no endpoints in common (see Figure 15). The only if part of the next theorem can be found in [2]. The theorem also follows from Theorem 5.

Theorem 8. *A cyclic quadrilateral of side lengths a, b, c, d is circumscribing a circle if and only if*

$$\sqrt{\bar{a}}+\sqrt{\bar{c}} = \sqrt{\bar{b}}+\sqrt{\bar{d}}.$$

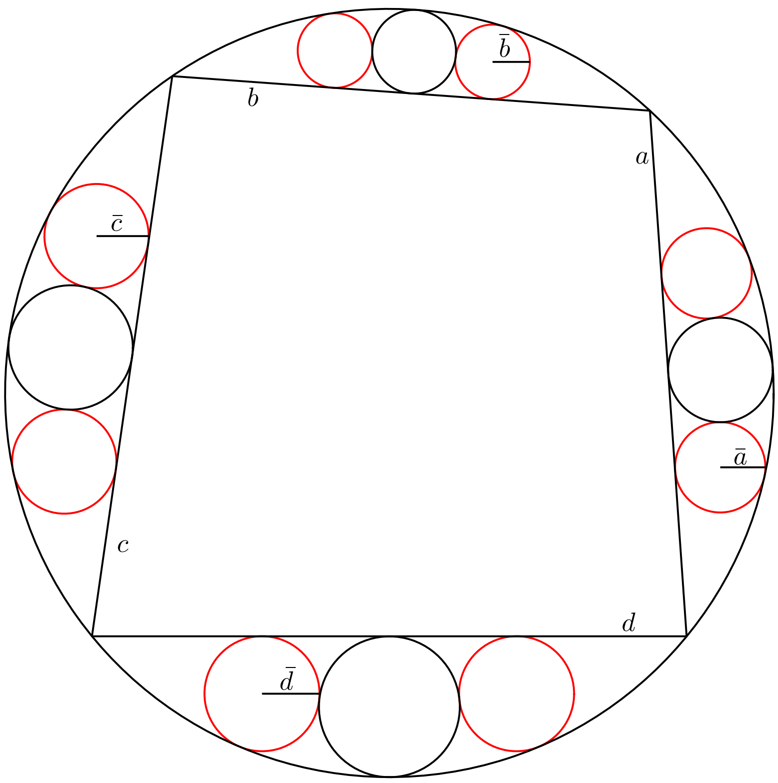

Figure 15.

The next theorem can also be found in [1].

Theorem 9. *For a cyclic quadrilateral of side lengths a, b, c, d and circumradius R, we have*

$$R = \frac{16(\sqrt{\bar{a}\bar{b}} + \sqrt{\bar{c}\bar{d}})(\sqrt{\bar{b}\bar{c}} + \sqrt{\bar{d}\bar{a}})(\sqrt{\bar{a}\bar{c}} + \sqrt{\bar{b}\bar{d}})}{(s - 2\sqrt{\bar{a}})(s - 2\sqrt{\bar{b}})(s - 2\sqrt{\bar{c}})(s - 2\sqrt{\bar{d}})}.$$

where $s = \sqrt{\bar{a}} + \sqrt{\bar{b}} + \sqrt{\bar{c}} + \sqrt{\bar{d}}$.

Proof. The theorem is proved in a similar way as Theorem 7 using Parameshvara's formula:

$$R = \sqrt{\frac{(ab + cd)(ac + bd)(ad + bc)}{(-a + b + c + d)(a - b + c + d)(a + b - c + d)(a + b + c - d)}}.$$

□

The next corollary can be found in [8] and also was given in a sangaku hung in 1865 [7].

Corollary 3. *For an isosceles trapezoid of side lengths a, b, c, d with $b = d$ and circumradius R, we have*

$$R = \frac{16\bar{b}(\bar{b} + \sqrt{\bar{a}\bar{c}})}{4\bar{b} - (\sqrt{\bar{a}} - \sqrt{\bar{c}})^2}.$$

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References

[1] Aida (会田安明) ed., Sampō Tenshōhō (算法天生法) 1788, Tohoku University Digital Collection.

[2] Baba (馬場正統) ed., Jimon Jitoh Daijutsu (自問自答題術) 1822, Tohoku University Digital Collection.

[3] Q. T. Bui, A Newly Born Pair of Siblings to Archimedes' Twins, (November 12 2011), https://www.cut-the-knot.org/Curriculum/Geometry/ArbelosBui.shtml.

[4] C. W. Dodge, T. Schoch, P. W. Woo and P. Yiu, Those ubiquitous Archimedean circles, Math. Mag., **72**(3) (1999) 202–213.

[5] Fujita (藤田嘉言) ed., Simpeki Sampō (or Sampoh, Sanpou, Sanpoh) (神壁算法) 1789.

[6] A. Hirayama, A. Mitsuhashi, M. Ohno, The sangaku in Chiba prefecture (千葉県の算額), Naritasan Shiryōkan (成田山資料館), 1970.

[7] Kamoshita (鴨下寅吉) ed., Kokufu Sangaku (国府算額), transcripted in 1872, Tohoku University Digital Collection.

[8] Kawakita (川北朝鄰) ed., Sampō Senjutsu (算法浅述) 1864, Tohoku University Digital Collection.

[9] H. Okumura, Archimedean twin circles in the arbelos, Math. Gaz., **97** (2013) 512.

[10] H. Okumura and M. Watanabe, The Archimedean circles of Schoch and Woo, Forum Geom., **4** (2004) 27–34.

[11] Saitama prefectural library (埼玉県立図書館) ed., The Sangaku in Saitama (埼玉の算額), 1969, Saitama Prefectural Library (埼玉県立図書館).

[12] S. Takagi, The sangaku in Gifu (岐阜県の算額の解説), 1986, private publication.

[13] Toyoyoshi (豊由周齋), Tenzan (點竄), Department of Mathematics Kyoto University Digital Library.

[14] Kururisha Sandai Shu (久留里社算題集), Tohoku University Digital Collection.

[15] Problems 2019-3, Sangaku J. Math., **3** (2019) 26–28.

[16] Problems 2017-3, Sangaku J. Math., **1** (2017) 21–23.