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A generalization of Problem 2019-4 and division by zero

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Abstract. We give a simple solution of Problem 2019-4, and give a generalization of the problem using division by zero. A variation of the problem is also considered.

Keywords. incircle and excircles, triangle with parallel sides, division by zero.

Mathematics Subject Classification (2010). 01A27, 51M04.

1. Introduction

In this article we give a simple solution of Problem 2019-4 and give a generalization of the problem with division by zero, and give several related results. The problem is stated as follows (see Figure 1):

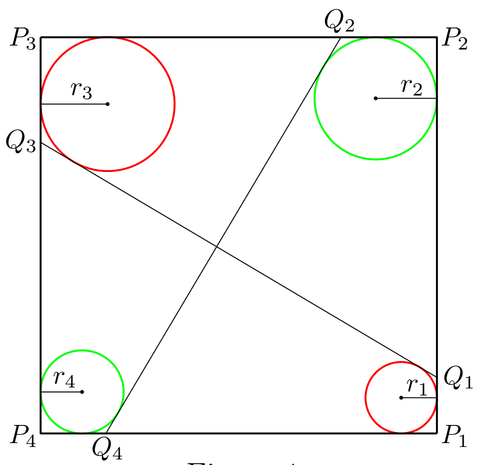

Figure 1.

Problem 1 ([6])**.** For a square $P_1P_2P_3P_4$, let Q_i $(i = 1, 2, 3, 4)$ be a point on the side P_iP_{i+1} such that $Q_1Q_3 \perp Q_2Q_4$, where the subscripts are taken modulo 4. Let r_i be the radius of the circle lying inside of the square and touching $P_iP_{i\pm1}$ and Q_iQ_{i+2}. Prove or disprove $r_1 + r_3 = r_2 + r_4$.

2. Generalization

In this section we generalize Problem 1. Let Q_i be a point on the segment P_iP_{i+1} for a square $P_1P_2P_3P_4$ such that $Q_1Q_3 \perp Q_2Q_4$ and assume that the line Q_1Q_3 meets the lines P_4P_1 and P_2P_3 in points R_1 and R_3, respectively. If the points P_1 and R_1 lie on the opposite side of P_3P_4, we consider by interchanging the symbols P_1 and P_4; P_2 and P_3; Q_1 and Q_3, respectively. Therefore we can always assume that P_1 and R_1 lie on the same side of P_3P_4. Let $a = |P_1P_2|$ and $\theta = \angle P_1R_1Q_1$, and assume that the line Q_2Q_4 meets the lines P_1P_2 and P_3P_4 in points R_2 and R_4, respectively. Then the triangles $P_iQ_iR_i$ $(i = 1, 2, 3, 4)$ are similar. Problem 1 and the result in [1] are generalized as follows:

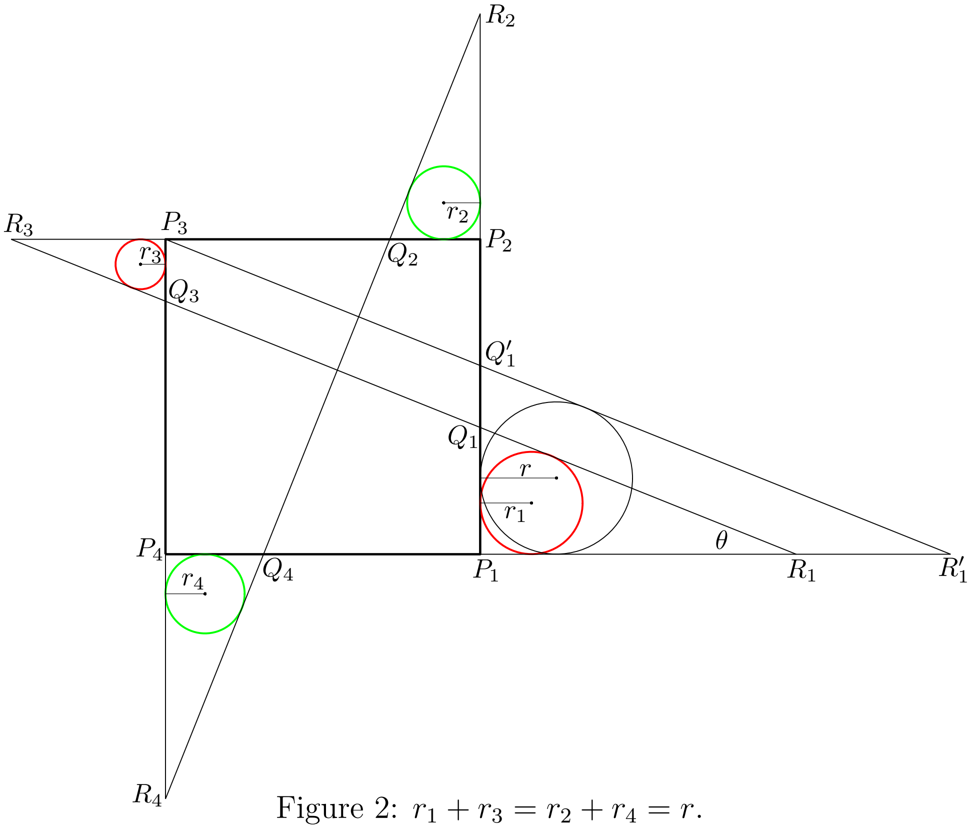

Figure 2: $r_1 + r_3 = r_2 + r_4 = r$.

Theorem 1. *The following statements hold:*
(i) *If r_i is the inradius of the triangle $P_iQ_iR_i$, then*

$$\frac{1}{r_1 + r_3} = \frac{1}{r_2 + r_4} = \frac{1 + \cos\theta + \sin\theta}{a(\cos\theta - \sin\theta)}.$$

(ii) *If r_i^p is the radius of the excircle of $P_iQ_iR_i$ touching Q_iR_i from the side opposite to P_i, then*

$$\frac{1}{r_1^p + r_3^p} = \frac{1}{r_2^p + r_4^p} = \frac{-1 + \cos\theta + \sin\theta}{a(\cos\theta - \sin\theta)}.$$

(iii) *If r_i^q is the radius of the excircle of $P_iQ_iR_i$ touching R_iP_i from the side opposite to Q_i, then*

$$\frac{1}{r_1^q + r_3^q} = \frac{1}{r_2^q + r_4^q} = \frac{1}{a}\left(-1 + \frac{1}{\cos\theta - \sin\theta}\right).$$

(iv) *If r_i^r is the radius of the excircle of $P_iQ_iR_i$ touching P_iQ_i from the side opposite to R_i, then*

$$\frac{1}{r_1^r + r_3^r} = \frac{1}{r_2^r + r_4^r} = \frac{1}{a}\left(1 + \frac{1}{\cos\theta - \sin\theta}\right).$$

Proof. We assume that the line parallel to Q_1Q_3 passing through P_3 meets P_1P_2 and P_4P_1 in points Q_1' and R_1', respectively, and r is the inradius of the triangle $P_1Q_1'R_1'$ (see Figure 2). Then the triangles $P_1Q_1R_1$, $P_3Q_3R_3$ and $P_1Q_1'R_1'$ are similar. While $|Q_1Q_1'| = |P_3Q_3|$ implies the relation $|P_1Q_1| + |P_3Q_3| = |P_1Q_1'|$. Therefore we get $r_1 + r_3 = r$. Similarly we get $r_2 + r_4 = r$. From the equations

$$\frac{|P_3P_4|}{|P_4R_1'|} = \frac{a}{a + r + r\cot\frac{\theta}{2}} = \tan\theta,$$

we get the following equations, i.e., (i) is proved.

$$r = \frac{-1 + \cot\theta}{1 + \cot\frac{\theta}{2}}a = \frac{\cos\theta - \sin\theta}{1 + \cos\theta + \sin\theta}a.$$

Let r^p be the radius of the excircle of $P_1Q_1'R_1'$ touching $Q_1'R_1'$ from the side opposite to P_1 (see Figure 3). Then we get $r^p = r_1^p + r_3^p = r_2^p + r_4^p$ by (i). From the equations

$$\frac{|P_3P_4|}{|P_4R_1'|} = \frac{a}{a + r^p - r^p\tan\frac{\theta}{2}} = \tan\theta,$$

we get the following equations, i.e., (ii) is proved.

$$r^p = \frac{-1 + \cot\theta}{1 - \tan\frac{\theta}{2}}a = \frac{\cos\theta - \sin\theta}{-1 + \cos\theta + \sin\theta}a.$$

Let r^q be the radius of the excircle of $P_1Q_1'R_1'$ touching $R_1'P_1$ from the side opposite to Q_1'. Then $r^q = r_1^q + r_3^q = r_2^q + r_4^q$ by (i). From the equations

$$\frac{|P_3P_4|}{|P_4R_1'|} = \frac{a}{a + r^q + r^q\tan\frac{\theta}{2}} = \tan\theta,$$

we get the following equations, i.e., (iii) is proved.

$$r^q = \frac{-1 + \cot\theta}{1 + \tan\frac{\theta}{2}}a = \frac{\cos\theta - \sin\theta}{1 - \cos\theta + \sin\theta}a.$$

Let r^r be the radius of the excircle of $P_1Q_1'R_1'$ touching P_1Q_1' from the side opposite to R_1'. Then $r^r = r_1^r + r_3^r = r_2^r + r_4^r$ by (i). From the equations

$$\frac{|P_3P_4|}{|P_4R_1'|} = \frac{a}{a - r^r + r^r\cot\frac{\theta}{2}} = \tan\theta,$$

we get the following equations, i.e., (iv) is proved.

$$r^r = \frac{-1 + \cot\theta}{-1 + \cot\frac{\theta}{2}}a = \frac{\cos\theta - \sin\theta}{1 + \cos\theta - \sin\theta}a.$$

□

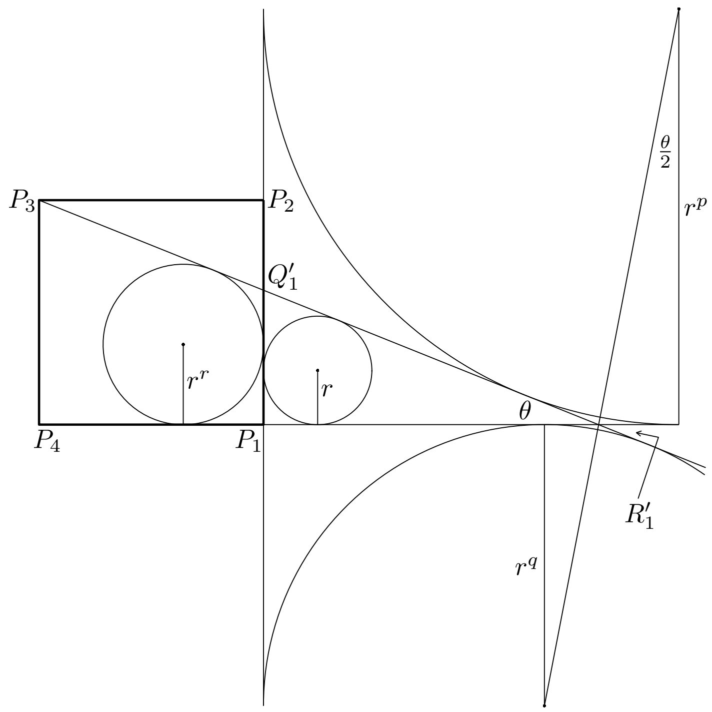

Figure 3.

Corollary 1. *Let r_i^q, r_i^r be as in Theorem 1. The following relation holds.*

$$\frac{1}{r_i^q + r_{i+2}^q} - \frac{1}{r_i^r + r_{i+2}^r} = -\frac{2}{a}.$$

3. A variation of Problem 1

Let P_i, Q_i, R_i be the points as in the previous section under the assumption that the point R_1 lies on the same side of P_3P_4 as P_1. In the previous section we have considered the incircle and the excircles of the triangle $P_iQ_iR_i$. In this section we consider the incircle and the excircles of the triangle $P_iQ_{i-1}R_{i+1}$ (see Figure 4).

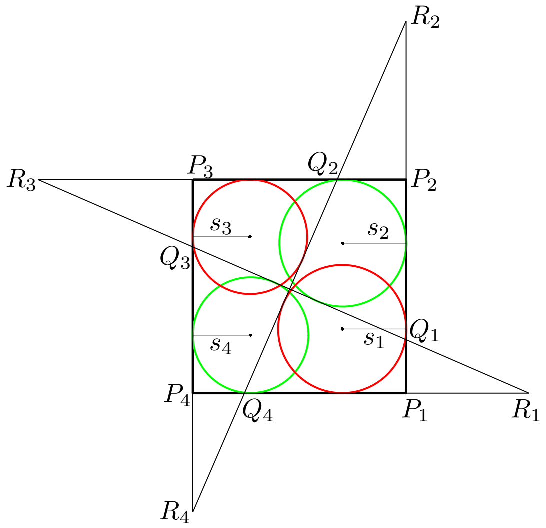

Figure 4: $s_1 + s_3 = s_2 + s_4$.

Theorem 2. *The following statements hold.*
(i) *If s_i is the inradius of the triangle $P_iQ_{i-1}R_{i+1}$, the following relation holds.*

$$\frac{1}{s_1+s_3}=\frac{1}{s_2+s_4}=\frac{1}{a}\left(1+\frac{1}{\cos\theta+\sin\theta}\right).$$

(ii) *If s_i^p is the radius of the excircle of $P_iQ_{i-1}R_{i+1}$ touching $Q_{i-1}R_{i+1}$ from the side opposite to P_i, then the following relation holds.*

$$\frac{1}{s_1^p+s_3^p}=\frac{1}{s_2^p+s_4^p}=\frac{1}{a}\left(1-\frac{1}{\cos\theta+\sin\theta}\right).$$

(iii) *If s_i^q is the radius of the excircle of $P_iQ_{i-1}R_{i+1}$ touching $R_{i+1}P_i$ from the side opposite to Q_{i-1}, then the following relation holds.*

$$\frac{1}{s_1^q+s_3^q}=\frac{1}{s_2^q+s_4^q}=\frac{1-\cos\theta+\sin\theta}{a(\cos\theta+\sin\theta)}.$$

(iv) *If s_i^r is the radius of the excircle $P_iQ_{i-1}R_{i+1}$ touching P_iQ_{i-1} from the side opposite to R_{i+1}, then the following relation holds.*

$$\frac{1}{s_1^r+s_3^r}=\frac{1}{s_2^r+s_4^r}=\frac{1+\cos\theta-\sin\theta}{a(\cos\theta+\sin\theta)}.$$

Proof. We assume that the line parallel to Q_1Q_3 passing through P_2 meets the lines P_4P_1 and P_3P_4 in points R_1' and Q_3', respectively, and the line parallel to P_2P_3 passing through Q_3 meets $R_1'Q_3'$ in a point S (see Figure 5). Let s be the inradius of the triangle $P_4Q_3'R_1'$. Then the triangles $P_2Q_1R_3$ and $Q_3Q_3'S$ are congruent. Therefore s_2 equals the inradius of $Q_3Q_3'S$. While the triangles $Q_3Q_3'S$, $P_4Q_3R_1$ and $P_4Q_3'R_1'$ are similar and $|Q_3Q_3'|+|P_4Q_3|=|P_4Q_3'|$. Therefore $s_2+s_4=s$. Similarly we get $s_1+s_3=s$. On the other hand, from the equations

$$\frac{|P_1P_2|}{|P_1R_1'|}=\frac{a}{s+s\cot\frac{\theta}{2}-a}=\tan\theta,$$

we get the following equations, i.e., (i) is proved.

$$s=\frac{1+\cot\theta}{1+\cot\frac{\theta}{2}}a=\frac{(\cos\theta+\sin\theta)a}{1+\cos\theta+\sin\theta}.$$

Let s^p be the radius of the excircles of $P_4Q_3'R_1'$ touching $Q_3'R_1'$ from the side opposite to P_4, then we have $s^p=s_1^p+s_3^p=s_2^p+s_4^p$ by (i). From the equations

$$\frac{|P_1P_2|}{|P_1R_1'|}=\frac{a}{s^p-s^p\tan\frac{\theta}{2}-a}=\tan\theta,$$

we get the following equations, i.e., (ii) is proved.

$$s^p=\frac{1+\cot\theta}{1-\tan\frac{\theta}{2}}a=\frac{(\cos\theta+\sin\theta)a}{-1+\cos\theta+\sin\theta}.$$

Let s^q be the radius of the excircles of $P_4Q_3'R_1'$ touching P_4R_1' from the side opposite to Q_3'. Then we have $s^q=s_1^q+s_3^q=s_2^q+s_4^q$ by (i). From the equations

$$\frac{|P_1P_2|}{|P_1R_1'|}=\frac{a}{s^q+s^q\tan\frac{\theta}{2}-a}=\tan\theta,$$

we get the following equations, i.e., (iii) is proved.

$$s^q = \frac{1+\cot\theta}{1+\tan\frac{\theta}{2}}a = \frac{(\cos\theta+\sin\theta)a}{1-\cos\theta+\sin\theta}.$$

Let s^r be the radius of the excircles of $P_4Q_3'R_1'$ touching P_4Q_3' from the side opposite to R_1'. Then we have $s^r = s_1^r + s_3^r = s_2^r + s_4^r$ by (i). From the equations

$$\frac{|P_1P_2|}{|P_1R_1'|} = \frac{a}{-s^r + s^r\cot\frac{\theta}{2} - a} = \tan\frac{\theta}{2},$$

we get the following equations, i.e., (iv) is proved.

$$s^r = \frac{1+\cot\theta}{-1+\cot\frac{\theta}{2}}a = \frac{(\cos\theta+\sin\theta)a}{1+\cos\theta-\sin\theta}.$$

□

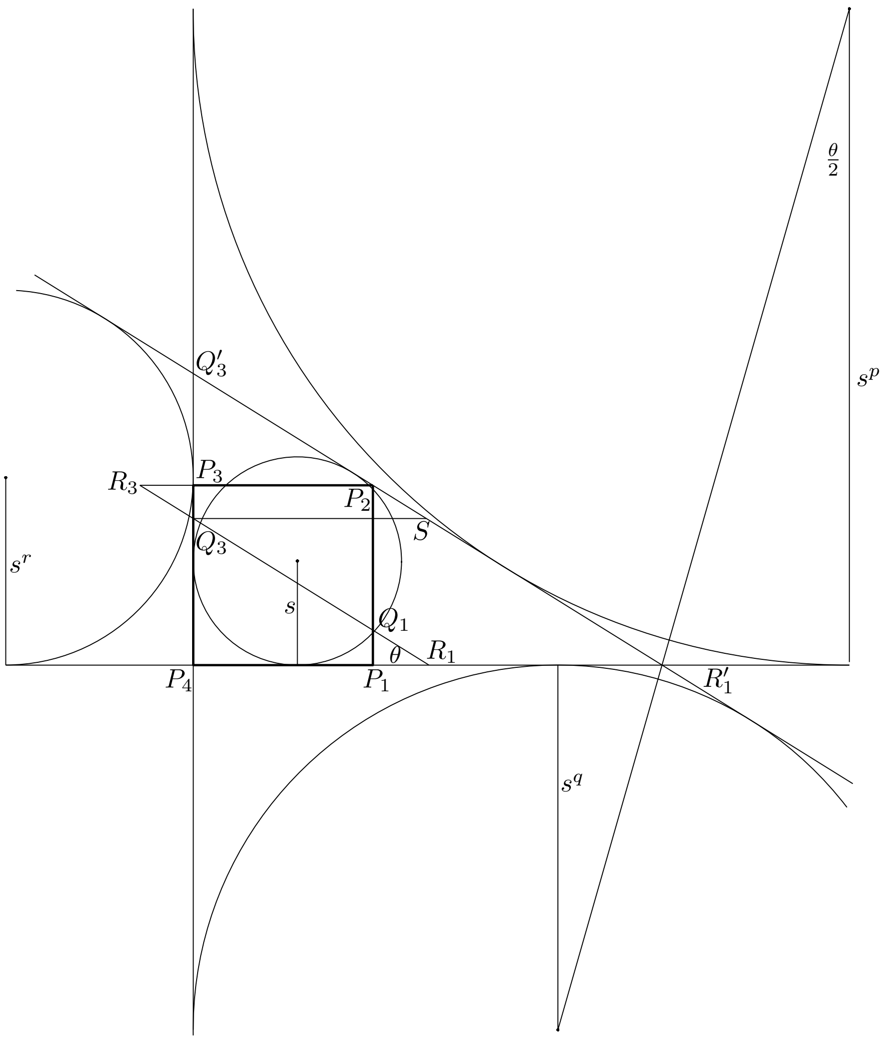

Figure 5.

The parts (i) and (ii) in the Theorem 2 can be found in [3] with a proof and in [4] with no proof. The proof in [3] is different from the above proof. Next corollary can also be found in [3, 4].

Corollary 2. *Let s_i, s_i^p be as in Theorem 2. The following relation holds.*

$$\frac{1}{s_i + s_{i+2}} + \frac{1}{s_i^p + s_{i+2}^p} = \frac{2}{a}.$$

By Corollaries 1 and 2, we get the next corollary.

Corollary 3. *Let r_i^q, r_i^r be as in Theorem 1, and let s_i, s_i^p be as in Theorem 2. The following relation holds.*

$$\frac{1}{r_i^q + r_{i+2}^q} + \frac{1}{s_i + s_{i+2}} + \frac{1}{s_i^p + s_{i+2}^p} = \frac{1}{r_i^r + r_{i+2}^r}.$$

4. Excircles of a triangle with parallel sides

In this section we show the next theorem by division be zero.

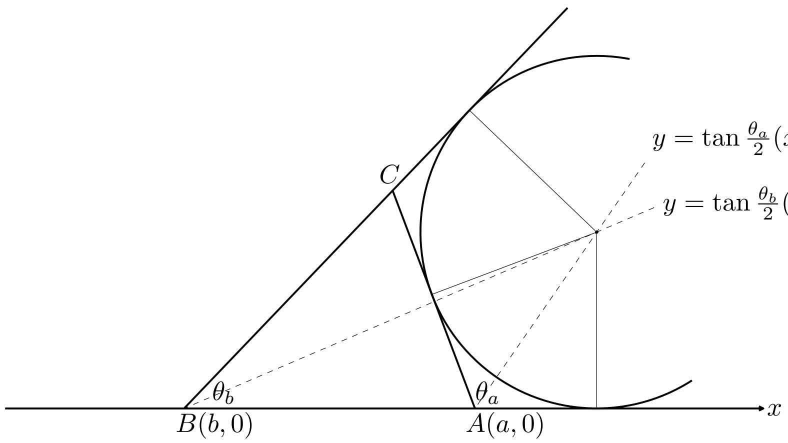

Figure 6.

Theorem 3. *For a triangle ABC, the radius of the excircle touching CA from the side opposite to B equals 0 if BC and CA are parallel.*

Proof. We use a rectangular coordinate system such that A and B have coordinates $(a, 0)$ and $(b, 0)$, respectively, where we assume that the point C lies on the region $y > 0$ (see Figure 6). Let θ_a (resp. θ_b) be the angles between $\overrightarrow{BA}$ and $\overrightarrow{AC}$ (resp. $\overrightarrow{BC}$). Then the center of the excircle coincides with the point of intersection of the two lines expressed by the equations $y = \tan\frac{\theta_a}{2}(x-a)$ and $y = \tan\frac{\theta_b}{2}(x-b)$. The coordinates of the point are give by

$$\left(\frac{a\tan\frac{\theta_a}{2} - b\tan\frac{\theta_b}{2}}{\tan\frac{\theta_a}{2} - \tan\frac{\theta_b}{2}}, (a-b)\frac{\sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}}{\sin\frac{\theta_a-\theta_b}{2}}\right),$$

where notice that the y-coordinate gives the exradius. Now we fix the points A, B and the angle θ_b and consider the case $\theta_a = \theta_b$ (see Figure 7). Then by the definition of division by zero [2],

$$\frac{z}{0} = 0 \text{ for any real number } z, \tag{1}$$

we see that the center of the excircle has coordinates $(0, 0)$, i.e., the exradius equals 0. $\square$

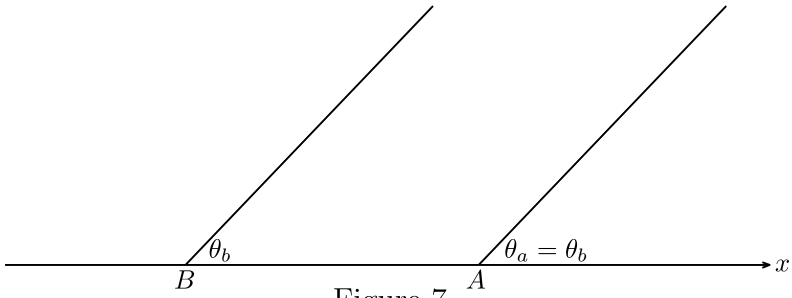

Figure 7.

Remark. The essential part of Theorem 3 is the fact that the point of intersection of two parallel lines coincides with the origin, which is pointed out in [5].

5. Parallel case

In this section we consider the case in which the lines P_4P_1 and Q_1Q_3 are parallel or $\theta = 0$ in Theorems 1 and 2 by Theorem 3. Let us assume $\theta = 0$. Then we get the relations

$$\frac{1}{r_1 + r_3} = \frac{1}{r_2 + r_4} = \frac{2}{a} \quad \text{and} \quad \frac{1}{r_1^r + r_3^r} = \frac{1}{r_2^r + r_4^r} = \frac{2}{a}.$$

Therefore (i) and (iv) in Theorem 1 are true. On the other hand we get $r_i^p = r_i^q = 0$ by Theorem 3. Therefore the equations in (ii) and (iii) are also true, because

$$\frac{1}{r_1^p + r_3^p} = \frac{1}{r_2^p + r_4^p} = 0 \quad \text{and} \quad \frac{-1 + \cos\theta + \sin\theta}{a(\cos\theta - \sin\theta)} = 0$$

and

$$\frac{1}{r_1^q + r_3^q} = \frac{1}{r_2^q + r_4^q} = 0 \quad \text{and} \quad \frac{1}{a}\left(-1 + \frac{1}{\cos\theta - \sin\theta}\right) = 0$$

Therefore (ii) and (iii) hold, i.e., Theorem 1 also holds in this case.

Let us consider Theorem 2 in the case $\theta = 0$. Then (i) and (iv) are true, since both sides of each of the equations equals $2/a$. On the other hand $s_i^p = s_i^q = 0$ by Theorem 3. Therefore we get

$$\frac{1}{s_1^p + s_3^p} = \frac{1}{s_2^p + s_4^p} = 0 \quad \text{while} \quad \frac{1}{a}\left(1 - \frac{1}{\cos\theta + \sin\theta}\right) = 0$$

and

$$\frac{1}{s_1^q + s_3^q} = \frac{1}{s_2^q + s_4^q} = 0 \quad \text{while} \quad \frac{1 - \cos\theta + \sin\theta}{a(\cos\theta + \sin\theta)} = 0.$$

Therefore (ii) and (iii) hold, i.e., Theorem 2 also holds in this case.

Notice that Corollaries 1, 2 and 3 also hold in thie case.

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