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Relationships Between Six Areas

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Abstract. If P is a point inside $\triangle ABC$, then the cevians through P divide $\triangle ABC$ into six small triangles. We give theorems about the relationships between the areas of these triangles.

Keywords. Triangles, cevians, areas, cevasix configuration.

Mathematics Subject Classification (2020). 51M04.

1. Introduction

Let P be any point inside a triangle ABC. The cevians through P divide $\triangle ABC$ into six smaller triangles, all having one vertex at P. The areas of these triangles will be named K_1 through K_6 as shown in Figure 1. The area of $\triangle ABC$ will be named K.

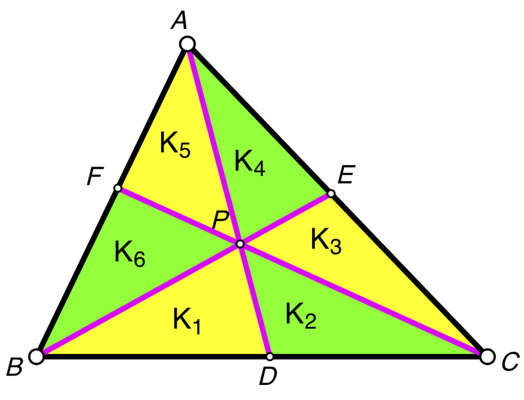

Figure 1. numbering of the six areas

In 1835, a wooden tablet was hung by Sugita Naotake in the Izanagi shrine in the Mie Prefecture of Japan that asked for a formula for K in terms of K_1, K_2, and K_3. See [1] and [2, p. 46]. The surprising answer is

$$K = \frac{K_2(K_1+K_2)(K_1+K_2+K_3)}{K_2(K_1+K_2+K_3)-K_3(K_1+K_2)},$$

or equivalently,

$$K = \frac{K_2(K_1+K_2)(K_1+K_2+K_3)}{K_2(K_1+K_2)-K_1K_3}.$$

We will call this Naotake's Formula. Proofs can be found in [5] and [3, pp. 210-212].

In this paper, we will find other relationships between these areas.

First, we mention two known results.

Theorem 1.1. *For any point P inside $\triangle ABC$, $K_1K_3K_5 = K_2K_4K_6$.*

A proof can be found in [6, Theorem 7.4].

Theorem 1.2. *For any point P inside $\triangle ABC$,*

$$\frac{1}{K_1}+\frac{1}{K_3}+\frac{1}{K_5}=\frac{1}{K_2}+\frac{1}{K_4}+\frac{1}{K_6}.$$

A proof can be found in [4, p. 43].

We now give some new results. Specifically, we will find formulas expressing each of K_4, K_5, and K_6 in terms of K_1, K_2, and K_3. We will give the results first. Then we will explain how we empirically came up with these results using a computer algebra system. Finally, we will explain how we proved the results.

2. The Results

We found the following results.

Theorem 2.1. *For any point P inside $\triangle ABC$,*

$$K_4 = \frac{K_1K_3^2}{K_2(K_1+K_2)-K_1K_3},$$

$$K_5 = \frac{K_1K_2K_3^2(K_1+K_2)}{(K_2(K_1+K_2)-K_1K_3)(K_2(K_1+K_2+K_3)-K_1K_3)},$$

$$K_6 = \frac{K_1K_3(K_1+K_2)}{K_2(K_1+K_2+K_3)-K_1K_3}.$$

Note that $K_5 = K_2K_4K_6/(K_1K_3)$ which agrees with Theorem 1.1.

3. How We Discovered the Results

Let us start by seeing how we discovered the formula for K_4.

Let $AB = c$, $BC = a$, and $CA = b$. Let the cevians through P be AD, BE, and CF. The twelve segments formed by them with each other and the sides of the triangle have lengths as shown in Figure 2.

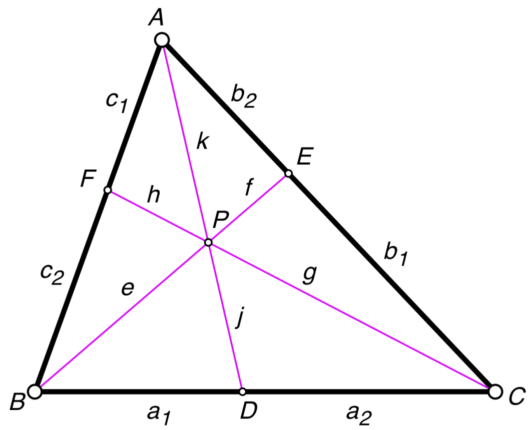

FIGURE 2. lengths of the twelve segments

We want to express each of the K_i in terms of a_1, a_2, b_1, b_2, c_1, c_2, and K. First, we will express e, f, g, h, j, and k in terms of a_1, a_2, b_1, b_2, c_1, and c_2.

The following result was stated by van Aubel in 1882 [7] and is often called Van Aubel's Theorem for Triangles [8].

Lemma 3.1 (Van Aubel's Theorem for Triangles)**.** *Let P be any point inside $\triangle ABC$ and let the cevians through P be AD, BE, and CF (Figure 3). Then*

$$\frac{AF}{FB}+\frac{AE}{EC}=\frac{AP}{PD}.$$

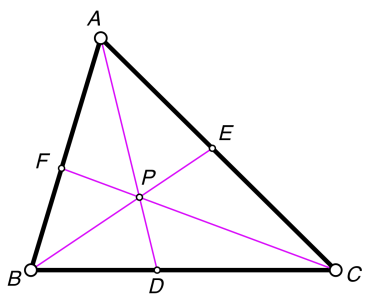

FIGURE 3.

An immediate consequence of Van Aubel's Theorem for Triangles is the following lemma.

Lemma 3.2. *Using the notation of Figure 2, we have the following equations.*

$$e=f\left(\frac{c_2}{c_1}+\frac{a_1}{a_2}\right),\quad g=h\left(\frac{b_1}{b_2}+\frac{a_2}{a_1}\right),\quad k=j\left(\frac{c_1}{c_2}+\frac{b_2}{b_1}\right).$$

Lemma 3.3. *The K_i can be expressed as multiples of K_1 using the lengths shown in Figure 2. In particular,*

$$K_2=\left(\frac{a_2}{a_1}\right)K_1,\quad K_3=\left(\frac{(a_1+a_2)f}{a_1e}\right)K_1,\quad K_4=\left(\frac{(a_1+a_2)b_2f}{a_1b_1e}\right)K_1,$$

$$K_5=\left(\frac{(a_1+a_2)(b_1+b_2)fh}{a_1b_1eg}\right)K_1,\quad K_6=\left(\frac{(a_1+a_2)(b_1+b_2)c_2fh}{a_1b_1c_1eg}\right)K_1.$$

Proof. If two triangles have the same altitude, then their areas are proportional to their bases. This gives the following proportions.

$$\frac{K_2}{K_1} = \frac{a_2}{a_1}, \quad \frac{K_3}{K_1+K_2} = \frac{f}{e}, \quad \frac{K_4}{K_3} = \frac{b_2}{b_1}, \quad \frac{K_5}{K_3+K_4} = \frac{h}{g}, \quad \frac{K_6}{K_5} = \frac{c_2}{c_1}.$$

Some algebraic manipulation (using Mathematica®) then gives us the desired equations. □

If we take Naotake's Formula,

$$K = \frac{K_2(K_1+K_2)(K_1+K_2+K_3)}{K_2(K_1+K_2+K_3) - K_3(K_1+K_2)},$$

and multiply both sides by the denominator and then bring all terms to the left side, we get the following relationship between K_1, K_2, K_3, and K.

$$KK_1K_2 - K_1^2K_2 + KK_2^2 - 2K_1K_2^2 - K_2^3 - KK_1K_3 - K_1K_2K_3 - K_2^2K_3 = 0$$

We observe that each term is of degree 3. It is expected that each term would have the same degree because the formula should remain the same if the triangle is scaled by a constant factor. So we note that this relationship is a homogeneous polynomial of degree 3. We also note that the coefficients are all small integers. This suggests how we could search for a formula relating K_1, K_2, K_3, and K_4.

Suppose we thought that the degree of the relationship is 2. We could then generate a list of all possible terms of degree 2 in four variables. The list would be

$$\{K_1^2,\ K_2^2,\ K_3^2,\ K_4^2,\ K_1K_2,\ K_1K_3,\ K_1K_4,\ K_2K_3,\ K_2K_4,\ K_3K_4\}.$$

Then we pick some fixed triangle with three random sides. We don't want a special triangle like an isosceles triangle or a right triangle. We pick numerical values of a_1, a_2, b_1, b_2, and c_1 for this triangle and pick c_2 so that Ceva's Theorem is satisfied, i.e. $a_1b_1c_1 = a_2b_2c_2$. We then use these values to find the values of K_1, K_2, K_3 and K_4 using Lemmas 3.2 and 3.3. We get expressions of the form A_1K_1, A_2K_1, A_3K_1 and A_4K_1, where A_1, A_2, A_3, and A_4 are strictly numerical. We can cancel the common factor of K_1 since the desired formula should not change when the triangle is scaled. We then look for a linear combination of A_1, A_2, A_3 and A_4 with small integer coefficients whose value is 0. The Mathematica® function `FindIntegerNullVector` can be used to possibly find such an integer linear combination. (This function uses a variant of the Lenstra-Lenstra-Lovasz lattice reduction algorithm.)

If a relationship, $f(A_1, A_2, A_3, A_4) = 0$ is found, then this is a good conjecture for what the relationship between K_1, K_2, K_3, and K_4 would be. We could also solve the polynomial equation $f(K_1, K_2, K_3, K_4) = 0$ for K_4 to get a conjectured formula for K_4 in terms of K_1, K_2, and K_3.

Using the `FindIntegerNullVector` function, no such linear combination was found.

We then guessed that the relationship might have degree 3. There are 20 terms of degree 3 in 4 variables. We repeated the procedure using our chosen numerical triangle and `FindIntegerNullVector` came up with the integer homogeneous linear combination

$$A_1A_2A_4 - A_1A_3A_4 - A_1A_3^2 + A_2^2A_4 = 0.$$

Solving for A_4 and changing the A's to K's then suggested that the formula we were looking for might be

$$K_4 = \frac{K_1K_3^2}{K_2(K_1+K_2)-K_1K_3}.$$

At this point, this is just a conjecture because the formula might only be valid for the particular numerical triangle we tested with, or the formula might only be accurate to the precision we used to perform the calculation. In order to mitigate erroneous results, we did all calculations to 25 decimal places of precision. Nevertheless, the formula found might still be an approximation. Before claiming the result as a theorem, we have to prove that the formula holds for all possible triangles and all positions of point P within that triangle.

4. How We Proved the Results

To prove the conjectured formula for K_4,

$$K_4 = \frac{K_1K_3^2}{K_2(K_1+K_2)-K_1K_3},$$

we proceed as follows. We form the symbolic expression

$$S = K_4 - \frac{K_1K_3^2}{K_2(K_1+K_2)-K_1K_3}.$$

We replace each K_i by the corresponding expression in terms of the symbolic variables a_1, a_2, b_1, b_2, and c_1 given by Lemma 3.3. Canceling out the common factor K_1, and after some simplification, we get

$$S' = \frac{(a_1+a_2)(a_1b_1f+a_1b_2f-a_2b_2e)f}{a_1b_1e(a_1f-a_2e)}.$$

Since these variables are arbitrary, they can generate all possible configurations, so it suffices to prove that S' is identically 0. To simplify S', we use the first equation of Lemma 3.2 to remove variable f. After simplification, we get

$$S' = \frac{(a_1+a_2)c_1(a_2b_2c_2-a_1b_1c_1)}{a_1b_1c_2(a_1c_1+a_2c_2)}.$$

Then it is clear that $S'=0$ by Ceva's Theorem. This proves the formula for K_4.

The formulas for K_5 and K_6 in terms of K_1, K_2, and K_3 were found and proved in the same manner and the details are omitted. The formula for K_5 could also be found by using Theorem 1.1 together with the formulas for K_4 and K_6.

The Wasan geometers did not have computers to help them with computations. A traditional proof of Naotake's Formula can be found in [3, pp. 210-212] and uses only elementary geometry. However, we do not know how Naotake came up with the formula in the first place. It certainly wasn't by the method used in this paper.

Open Question. *Is there a simpler method for discovering the formulas given in Theorem 2.1 without using computers?*

5. Expressing K in terms of K_1, K_3, and K_5

Naotake's Formula expresses K in terms of K_1, K_2, and K_3. It would be natural to look for formulas for K in terms of other triples of the K_i.

With the compute power available to me, the method described previously was unable to find a formula for K in terms of K_1, K_3, and K_5. However, we can find the relationship between K_1, K_3, K_5, and K, as follows. We can solve Naotake's formula for K_2, thereby expressing K_2 as a function of K_1, K_3, and K_5. Then we substitute this expression for K_2 into the formula for K_5 given by Theorem 2.1. This would give us a relationship between K_1, K_3, K_5, and K.

However, solving Naotake's formula for K_2 gives an expression involving radicals. We can avoid radicals by using the Mathematica® function `Eliminate`. Eliminating K_2 from the two equations gives the following result.

Theorem 5.1 (K in terms of K_1, K_3, and K_5)**.** *The areas K_1, K_3, K_5, and K satisfy the polynomial identity*

$$c_4K^4 - c_3K^3 - c_2K^2 - c_1K - c_0 = 0$$

where

$$\begin{aligned}
c_4 &= s_3, \\
c_3 &= s_1s_3, \\
c_2 &= 4s_2s_3 + e_1, \\
c_1 &= 2s_3^2 + s_1s_2s_3 + 2s_3e_2, \\
c_0 &= 2s_3^2s_1,
\end{aligned}$$

and where

$$\begin{aligned}
s_1 &= K_1 + K_3 + K_5, \\
s_2 &= K_1K_3 + K_3K_5 + K_5K_1, \\
s_3 &= K_1K_3K_5, \\
e_1 &= K_1^2K_3^3 + K_3^2K_5^3 + K_5^2K_1^3, \\
e_2 &= K_1K_3^2 + K_3K_5^2 + K_5K_1^2.
\end{aligned}$$

In theory, we could solve this equation for K using the quartic formula. This would give an explicit formula for K in terms of K_1, K_3, and K_5, but the resulting formula would contain a lot of radicals.

Equations relating K and other triples of the K_i can be found in a similar manner. We get the following.

Theorem 5.2 (K in terms of K_1, K_2, and K_4)**.** *For any point P inside $\triangle ABC$,*

$$K = \frac{(K_1+K_2)}{2K_1K_2} \times \left(K_1(2K_2+K_4) + \sqrt{K_1K_4(4K_1K_2 + K_1K_4 + 4K_2^2)} \right).$$

Theorem 5.3 (K in terms of K_1, K_3, and K_6)**.** *For any point P inside $\triangle ABC$,*

$$K = \frac{K_1}{2K_6} \times \frac{A + B\sqrt{C}}{K_1^2K_3 - K_6(K_1^2 + 2K_1K_3 - K_3K_6)}$$

where

$$A = K_1^2K_3^2 + K_6\left(K_3^2(K_6 - 2K_1) - K_1K_6(K_1 + K_6) - K_3K_6(K_1 + 3K_6)\right),$$
$$B = K_1(K_3 - K_6) - K_6(K_3 + K_6),$$
$$C = K_1^2(K_3 + K_6)^2 + K_3^2K_6^2 + 2K_1K_3K_6(3K_6 - K_3).$$

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