Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 5 (2021), pp. 39-42
Received 23 September 2021. Published on-line 27 September 2021
web: http://www.sangaku-journal.eu/

The arbelos in Wasan geometry: a problem in Sangaku Shōsen

Hiroshi Okumura
Takahanadai Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We consider a problem involving an arbelos in Sangaku Shōsen by division by zero.

Keywords. arbelos, constant product, division by zero.

Mathematics Subject Classification (2010). 01A27, 12E99, 51M04.

1. Introduction

For a point C on the segment AB such that $|BC| = 2a$, $|CA| = 2b$, we consider an arbelos formed by the two circles α, β of diameters BC, CA, respectively, and the semicircle γ of diameter AB (see Figure 1). In this article, we consider the following problem in [24] with division by zero $1/0 = 0$ [2], [23].

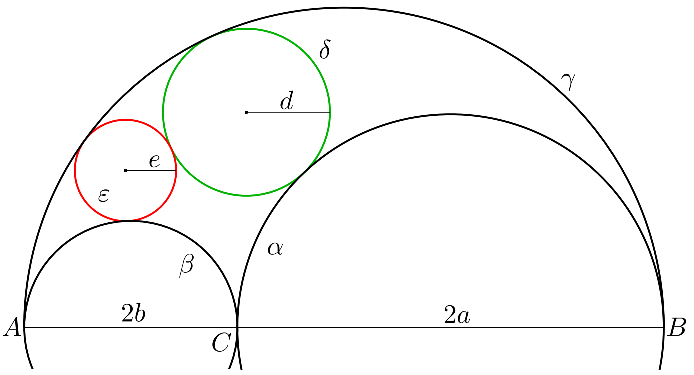

Figure 1.

Problem 1. Let δ (resp. ε) be a circle touching α (resp. β) and γ. If δ and ε touch externally and have radii d and e, respectively, show that the following relation holds:

$$e = \frac{a(a+b)^2(b-d)}{(a+b)^2 b - (a-b)^2 d}. \tag{1}$$

A solution of the problem can be found in [3]. There is an interesting special case for this problem in which the circle ε coincides with α or the circle δ coincides with β. By the symmetry of the figure, it is sufficient to consider one of the cases. In this paper we consider the case $\varepsilon = \alpha$.

2. Solution

In this section we give a solution of the problem. We use a rectangular coordinate system with origin C such that the farthest point on α from AB has coordinates (a, a). Let (x_d, y_d) and (x_e, y_e) be the coordinates of the centers of δ and ε, respectively. Since the center of the semicircle γ has x-coordinates $a - b$, we have the following five equations.

$$(x_d - a)^2 + y_d^2 = (a + d)^2,$$

$$(x_d - (a - b))^2 + y_d^2 = (a + b - d)^2,$$

$$(x_e + b)^2 + y_e^2 = (b + e)^2,$$

$$(x_e - (a - b))^2 + y_e^2 = (a + b - e)^2,$$

$$(x_d - x_e)^2 + (y_d - y_e)^2 = (d + e)^2.$$

Eliminating x_d, y_d, x_e, y_e from the equations, we get

$$\frac{ab}{ed} - \left(\frac{a}{e} + \frac{b}{d}\right) + \left(\frac{a-b}{a+b}\right)^2 = 0. \tag{2}$$

Solving the last equation for e, we get (1).

3. The case $\varepsilon = \alpha$

We now consider the case $\varepsilon = \alpha$. From now on, we assume division by zero $1/0 = 0$ [2], [23]. Then *a line has radius* 0 *as a circle and we can consider that two orthogonal figures touch each other* [23]. Therefore we consider that the line AB and the perpendicular to AB at B touch α and γ and have radius 0 as circles, i.e., the two figures are eligible to be the circle δ of radius 0.

3.1. **The case** $(e, d) = (a, 0)$**.** From (1), we get

$$a - e = \frac{4a^2b}{(a+b)^2b - (a-b)^2d}d.$$

Therefore $a = e$ implies $d = 0$. Therefore we can consider that $\delta = B$, or δ coincides with the line AB, or δ coincides with the perpendicular to AB at B in the case $\varepsilon = \alpha$. We denote the three figures by δ_1, δ_2 and δ_3, respectively (see Figure 2). If we consider a point or a line as a circle, the words "touch externally" in Problem 1 have no sense. Thereby we ignore the word "externally". Then $(\varepsilon, \delta) = (\alpha, \delta_i)$ satisfies the hypothesis of the problem for $i = 1, 2, 3$.

3.2. The case $(e,d)=(a,b)$. Notice that $p/q=0$ is equivalent to $p=0$ or $q=0$. From (2), we have

$$\left(\frac{a}{e}-1\right)\left(\frac{b}{d}-1\right)=\frac{4ab}{(a+b)^2}. \tag{3}$$

Therefore we get

$$\frac{a}{e}-1=\frac{4ab}{(a+b)^2}\Bigg/\left(\frac{b}{d}-1\right). \tag{4}$$

Assume $\varepsilon=\alpha$. Then the left side of (4) equals 0, i.e., the right side of (4) also equals 0. However the numerator of the right side does not equal 0. Hence the denominator equals 0, i.e., $d=b$. Therefore $\varepsilon=\alpha$ implies $\delta=\beta$ (see Figure 3).

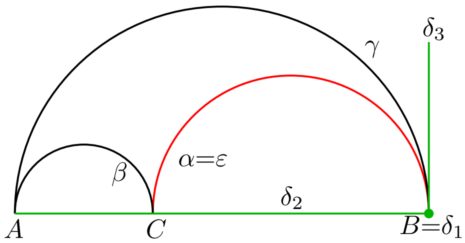

Figure 2.

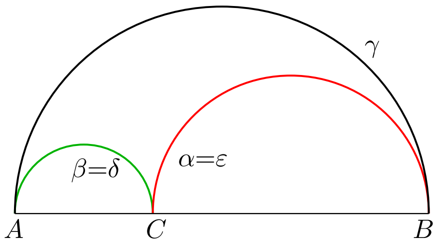

Figure 3.

The equation (3) shows that the value of the product in the left side is constant. While the equation (4) shows that $a/e-1$ being 0 is equivalent to $b/d-1$ being 0, which corresponds to the meaningful case described in Figure 3. However they cannot be 0 in (3). The fact shows that (4) is more useful than (3).

When the circle ε approaches to the circle α, the radius of the circle δ approaches to 0 and the circle approaches to the point B. When ε coincides with α, the circle δ moves instantaneously to the circle β and coincides with β in a discontinuous manner. Such a discontinuity can be found frequently when division by zero occurs. Consider the function $f(x)=1/x$ for example. We see $f(0)=0$ by $1/0=0$, while $\lim_{x\to+0}f(x)=+\infty$.

4. Conclusion

Division by zero $1/0=0$ was founded by Saburou Saitoh in 2014. As presented in this paper, it enables us to consider singular cases, which have never been considered before. Saburou Saitoh has been making a list of successful example applying division by zero and its generalization called division by zero calculus. There are more than 1200 examples in his list, which are evidences showing that an entirely new world of mathematics will be opened if we introduce them. For an extensive reference of division by zero and division by zero calculus including those evidences, see [23]. For applications to Euclidean spaces see [5]. For applications of division by zero calculus to plane geometry see [7]. Especially applications to Wasan geometry and circle geometry of division by zero and division by zero calculus, see [1], [4], [6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], [19, 20, 21, 22].

References

[1] Y. Kanai, H. Okumura, A three tangent congruent circle problem, Sangaku J. Math., **1** (2017) 16–20.

[2] M. Kuroda, H. Michiwaki, S. Saitoh, M. Yamane, New meanings of the division by zero and interpretations on $100/0 = 0$ and on $0/0 = 0$, Int. J. Appl. Math., **27**(2) (2014) 191–198.

[3] Matsunaga (松永直恒), Sampō Kaijutsu (算法解術) Volume 3, 1827, Tohoku University Digital Collection.

[4] T. Matsuura, H. Okumura, S. Saitoh, Division by zero calculus and Pompe's theorem, Sangaku J. Math., **3** (2019) 36–40.

[5] H. Michiwaki, H. Okumura, S. Saitoh, Division by Zero $z/0 = 0$ in Euclidean Spaces, Int. J. Math. Computation, **28**(1) (2017) 1–16.

[6] H. Okumura, Semicircles in the arbelos with overhang and division by zero, KoG, (2021), to appear.

[7] H. Okumura, Geometry and division by zero calculus, Int. J. Division By Zero Calculus, **1** (2021) 1–36.

[8] H. Okumura, The arbelos in Wasan geometry: Atsumi's problem with division by zero calculus, Sangaku J. Math., **5** (2021) 32–38.

[9] H. Okumura, Wasan Geometry. In: Sriraman B. (eds) Handbook of the Mathematics of the Arts and Sciences. Springer, Cham.

[10] H. Okumura, Solution to Problem 2017-1-6 with division by zero, Sangaku J. Math., **4** (2020) pp.73-76.

[11] H. Okumura, A generalization of Problem 2019-4 and division by zero, Sangaku J. Math., **4** (2020) 45–52.

[12] H. Okumura, A four circle problem and division by zero, Sangaku J. Math., **4** (2020) pp.1-8.

[13] H. Okumura, Remarks on Archimedean circles of Nagata and Ootoba, Sangaku J. Math., **3** (2019) 119–122.

[14] H. Okumura, The arbelos in Wasan geometry: Ootoba's problem and Archimedean circles, Sangaku J. Math.,**3** (2019) 91–97.

[15] H. Okumura, Is it really impossible to divide by zero?, Biostatistics and Biometrics Open Acc J., **7**(1) (2018) 1–2.

[16] H. Okumura, An analogue of Pappus chain theorem with division by zero, Forum Geom., **18** (2018) 409–412.

[17] H. Okumura, Solution to 2017-1 Problem 4 with division by zero, Sangaku J. Math., **2** (2018) 27–30.

[18] H. Okumura, Wasan geometry with the division by 0, Int. J. Geom., **7**(1) (2018) 17–20.

[19] H. Okumura, S. Saitoh, Wasan geometry and division by zero calculus, Sangaku J. Math., **2** (2018) 57–73.

[20] H. Okumura and S. Saitoh, Applications of the division by zero calculus to Wasan geometry, Glob. J. Adv. Res. Class. Mod. Geom., **7**(2) (2018) 44–49.

[21] H. Okumura and S. Saitoh, Harmonic mean and division by zero, Forum Geom., **18** (2018) 155–159.

[22] H. Okumura and S. Saitoh, Remarks for The twin circles of Archimedes in a skewed arbelos by Okumura and Watanabe, Forum Geom., **18** (2018) 97–100.

[23] S. Saitoh, Introduction to the Division by Zero Calculus, Scientific Research Publishing, Inc. 2021.

[24] Ushijima (牛島盛庸), Sangaku[2] Shōsen (算学小筌), 1794, Tohoku University Digital Collection.

[2]This does not mean a votive mathematical tablet, but mathematics itself.