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A note on circles touching two circles in a Pappus chain

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Abstract. We generalize the results for the circles touching two circles in a Pappus chain in [8].

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1. Introduction

For a square $ABCD$ of side length s, β is the semicircle of diameter AB lying inside of $ABCD$, γ is the image of β by the homothety with center A and ratio 2, $\alpha_1, \alpha_2, \alpha_3, \cdots$ is the Pappus chain touching β and γ such that α_1 touches the line AB (see Figure 1). There is a circle δ_n $(n \geq 1)$ of radius d_n touching α_n and α_{n+1} at their point of tangency and AB at A, where δ_0 is the line AB. Then we have [8]:

$$d_n = \frac{s}{n}. \tag{1}$$

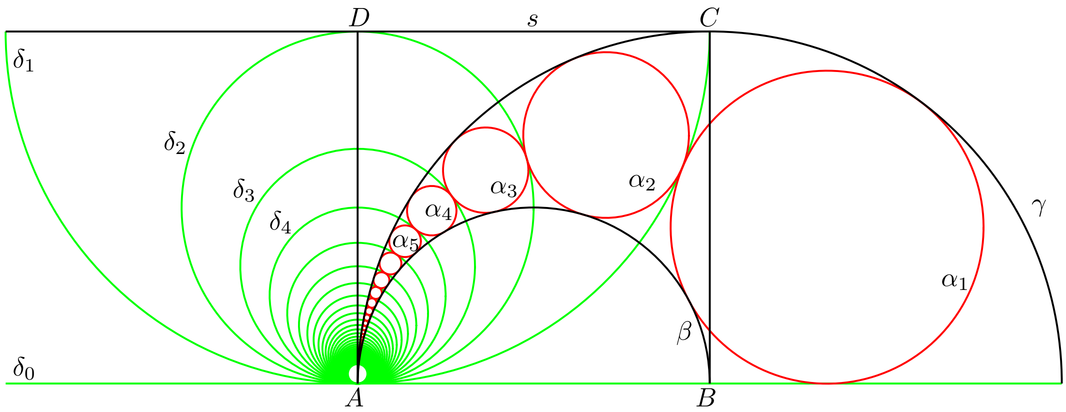

Figure 1.

The relation (1) is obtained by considering a sangaku problem in [17]. In this note we generalize this.

2. Division by zero

Since we use division by zero, we briefly introduce this. For a field F, we consider a canonical bijection $\psi : F \to F$ such that

$$\psi(a) = \begin{cases} a^{-1} & \text{if } a \neq 0\ , \\ a & \text{if } a = 0. \end{cases}$$

With this bijection, we define a noncommutative binary operation / or $\div$ on F as follows:

$$a/b = a \div b \stackrel{\text{def}}{=} a \cdot \psi(b) \quad for\ a, b \in F.$$

Indeed, this has been used in our mathematics and is called division, and a/b is also represented by $\frac{a}{b}$, which is called a fraction. However there has been a quaint custom not to consider $a/0 = a \cdot \psi(0) = a \cdot 0 = 0$. Therefore it is worthy to consider including this case, i.e., in this paper we assume

$$\frac{a}{0} = 0 \quad for\ \forall a \in F. \tag{2}$$

Notice that the cancellation law using 0 and the reduction of fraction can not be made for the fraction with 0 denominator.

Let $F = \mathbf{R}$. Any circle in the x-y plane has an equation

$$e(x^2 + y^2) - 2fx - 2gy + h = 0, \tag{3}$$

and has radius

$$\sqrt{\frac{f^2 + g^2 - eh}{e^2}}. \tag{4}$$

While (3) represents a line in the case $e = 0$, and (4) equals 0 by (2) in the same case. Therefore *a line can be considered as a circle of radius 0* [5], [16].

3. Generalization

In this section, we generalize (1). For a segment AB, let C be a point on it such that $|AC| = 2b$, $|BC| = 2a$ and $c = a + b$. The semicircles of diameters AC and AB constructed on the same side of AB are denoted by β and γ, respectively. Let α_1, α_2, α_3, $\cdots$ be the Pappus chain touching β and γ such that α_1 touches the segment AB (see Figure 2). If we invert the figure in the circle of center A orthogonal to α_n, the images of α_1, α_2, α_3, $\cdots$ are the circles congruent to α_n and their centers lie on the perpendicular from O_{α_n} to AB, where O_{α_n} is the center of the circle α_n. Therefore there are circles δ_1, δ_2, δ_3, $\cdots$ such that δ_i touches α_i and α_{i+1} at their point of tangency and AB at A, where we define that δ_0 is the line AB. Let d_n be the radius of δ_n. The next theorem is a generalization of (1).

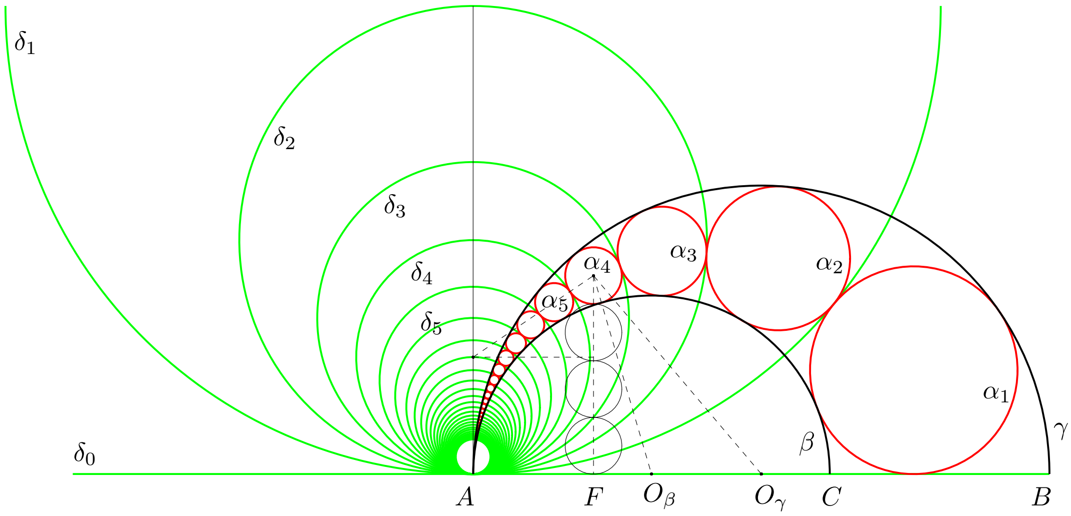

Figure 2: $n = 4$.

Theorem 1. *For a non-negative integer n, we have*

$$d_n = \frac{bc}{an}. \tag{5}$$

Proof. We use a rectangular coordinates system with origin A so that the farthest point on β from AB has coordinates (b, b). Assume that α_n has radius a_n and O_{α_n} has coordinates (p, q) for $n \geq 1$. Inverting $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the circle of center A orthogonal to α_n, we get

$$q = (2n-1)a_n. \tag{6}$$

Let F be the foot of perpendicular from O_{α_n} to AB. From the right triangles $FO_\beta O_{\alpha_n}$ and $FO_\gamma O_{\alpha_n}$ we have

$$q^2 + (b-p)^2 = (b+a_n)^2, \tag{7}$$

$$q^2 + (c-p)^2 = (c-a_n)^2. \tag{8}$$

Solving (7) and (8) for p and a_n with (6), we have

$$p = \frac{4bc(b+c)}{4bc + a^2(2n-1)^2}, \quad a_n = \frac{4abc}{4bc + a^2(2n-1)^2}. \tag{9}$$

From the right triangle formed by the segments FO_{α_n}, $O_{\alpha_n}O_{\delta_n}$ and the line passing through O_{δ_n} parallel to AB, we get

$$p^2 + (q - d_n)^2 = (a_n + d_n)^2.$$

Solving the last equation for d_n using (6) and (9), we have (5). If $n = 0$, (5) also holds, since a line can be considered as a circle of radius 0 and $z/0 = 0$ for any real number z as stated in section 2. □

The relations (9) is a generalization of Theorem 1 in [8]. Notice that if C is the midpoint of AB, i.e., if $b = a$, then the circles δ_1 and δ_2 have the same radii as to γ and β, respectively (see Figure 1). If $b > a$, then $d_1 = bc/a > c$ and $d_2 - b = bc/(2a) - b = b(c-2a)/(2a) = b(b-a)/(2a) > 0$. Therefore the circles δ_1 and δ_2 have radii larger than the radii of γ and β, respectively (see Figure 2). If $b < a$, then $d_1 = bc/a < c$ and $d_2 - b = b(b-a)/(2a) < 0$. Therefore the circles δ_1 and δ_2 have radii smaller than the radii of γ and β, respectively (see Figure 3).

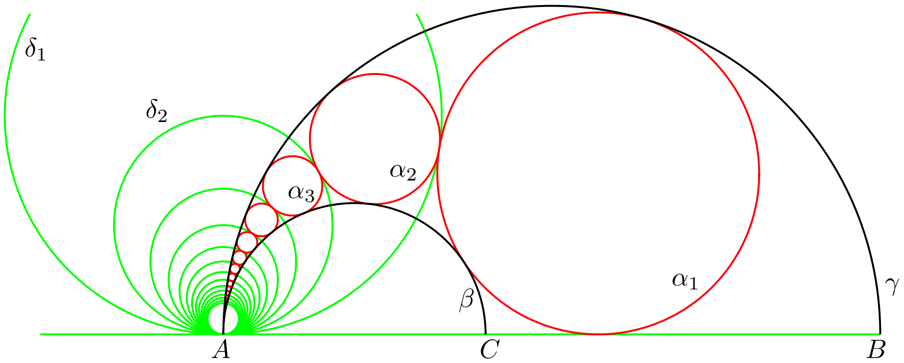

Figure 3.

Division by zero 1/0=0 and its generalization called division by zero calculus were founded by Saburou Saitoh [16]. Problems in Wasan geometry can easily be used to apply 1/0=0 and division by zero calculus. Such examples can be found in [1], [2], [3], [4], [6], [7, 8, 9, 10, 11, 12, 13, 14], [15].

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