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An Alternative Proof of the Japanese Theorem

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Abstract. We will prove the famous Japanese quadrilateral theorem and a related problem by applying a well-known sangaku problem.

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1. Introduction

The following sangaku problem is well known in Wasan geometry (See Figure 1). *Let D be a point on side BC of $\triangle ABC$, h be the distance from A to BC. $O_1(r_1)$, $O_2(r_2)$, $O(r)$ be the respective incircles of triangles ABD, ACD, and ABC. Then we have*

$$1-\frac{2r}{h}=\left(1-\frac{2r_1}{h}\right)\left(1-\frac{2r_2}{h}\right) \qquad \textit{or equivalently,} \quad r=r_1+r_2-\frac{2r_1r_2}{h}. \tag{1}$$

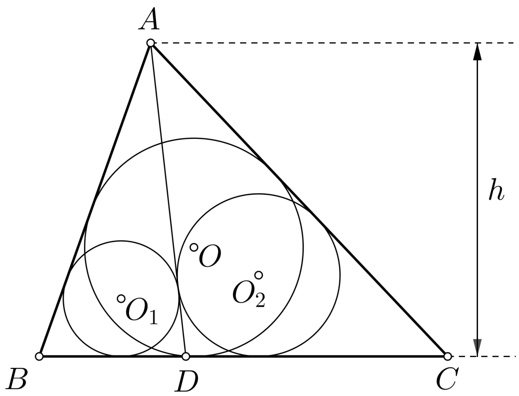

Figure 1

A proof can be found in [[1], pp. 33-34].

We will show an easy proof of the famous Japanese quadrilateral theorem that can be deduced from this problem. Also, a related problem proposed by Dr. Stanley Rabinowitz [2] can be solved by the same strategy. We will solve both problems.

2. PROOF OF THE JAPANESE QUADRILATERAL THEOREM

The Japanese quadrilateral theorem can be stated as follows (see Figure 2).

$A_1A_2A_3A_4$ is a cyclic quadrilateral. The circle $O_1(r_1)$ is inscribed in triangle $A_4A_1A_2$, the circle $O_2(r_2)$ is inscribed in triangle $A_1A_2A_3$, the circle $O_3(r_3)$ is inscribed in triangle $A_2A_3A_4$, and the circle $O_4(r_4)$ is inscribed in triangle $A_3A_4A_1$. Show that

$$r_1 + r_3 = r_2 + r_4.$$

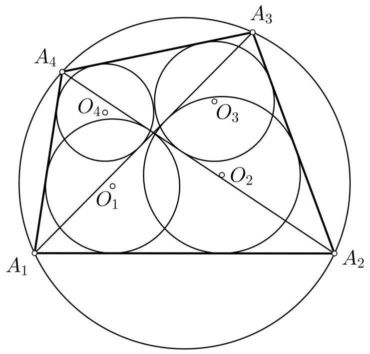

FIGURE 2

Proof. Assume $P = A_1A_3 \cap A_2A_4$. We will draw incircles of radii ρ_1, ρ_2, ρ_3, and ρ_4 of triangles A_1PA_4, A_2PA_1, A_3PA_2, and A_4PA_3, respectively (See figure 3). h_1, h_3 are the perpendiculars drawn from A_1 and A_3 on A_2A_4, respectively. Also, h_2, h_4 are the perpendiculars drawn from A_2 and A_4 on A_1A_3, respectively.

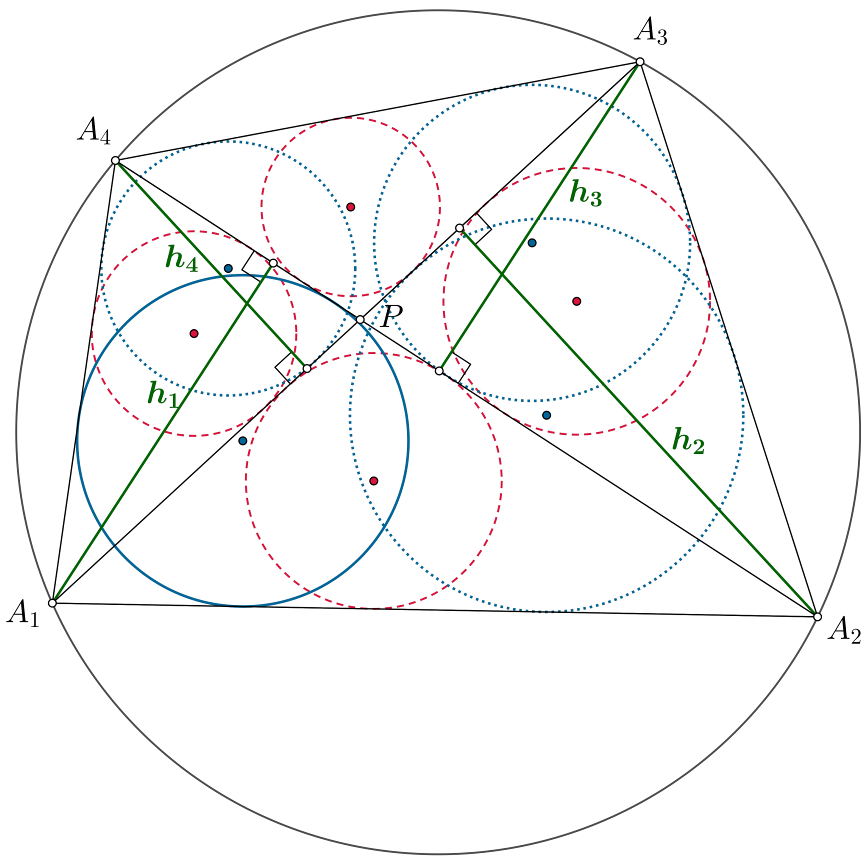

FIGURE 3

From similar triangles A_1PA_4 and A_2PA_3, we get

$$\frac{\rho_1}{h_1} = \frac{\rho_3}{h_2} \quad \text{which implies} \quad \frac{\rho_1\rho_2}{h_1} = \frac{\rho_2\rho_3}{h_2}, \tag{2}$$

and

$$\frac{\rho_1}{h_4} = \frac{\rho_3}{h_3} \quad \text{which implies} \quad \frac{\rho_4\rho_1}{h_4} = \frac{\rho_3\rho_4}{h_3}. \tag{3}$$

Applying (1) on triangles $A_4A_1A_2$, $A_1A_2A_3$, $A_2A_3A_4$ and $A_3A_4A_1$ successively, we get

$$r_1 = \rho_1 + \rho_2 - \frac{\rho_1\rho_2}{h_1}, \qquad r_2 = \rho_2 + \rho_3 - \frac{\rho_2\rho_3}{h_2},$$

$$r_3 = \rho_3 + \rho_4 - \frac{\rho_3\rho_4}{h_3}, \quad \text{and} \quad r_4 = \rho_4 + \rho_1 - \frac{\rho_4\rho_1}{h_4}.$$

Therefore, by virtue of (2) and (3), we get

$$r_1 - r_2 = \rho_1 - \rho_3 = r_4 - r_3 \quad \text{which implies} \quad r_1 + r_3 = r_2 + r_4. \quad \square$$

3. Solution to Dr. Rabinowitz's Problem

Dr. Rabinowitz's problem [2] can be stated as follows (see Figure 4). We will use the previous notations.

$A_1A_2A_3A_4$ is a cyclic quadrilateral. The circle $O_1(r_1)$ is inscribed in triangle $A_4A_1A_2$ and the circle $O_2(r_2)$ is inscribed in triangle $A_1A_2A_3$. If $P = A_1A_3 \cap A_2A_4$ and the circle $O_1'(\rho_1)$ is inscribed in triangle A_1PA_4 and circle $O_3'(\rho_3)$ is inscribed in triangle A_2PA_3. Show that

$$r_1 + \rho_3 = \rho_1 + r_2.$$

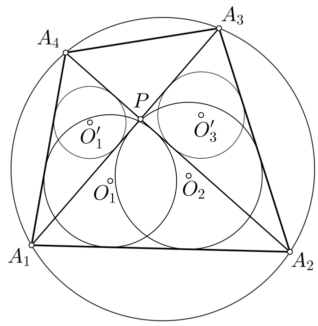

Figure 4

Proof. Observe that in the previous proof, the relation

$$r_1 - r_2 = \rho_1 - \rho_3$$

gives Dr. Rabinowitz's result. $\square$

References

[1] Hidetoshi Fukagawa and John Rigby, *Traditional Japanese Mathematics Problems of the 18th & 19th Centuries,* SCT Publishing, Singapore, 2002.

[2] https://www.facebook.com/photo/?fbid=10224010870491576&set=gm.4340635776050095