Sangaku Journal of Mathematics (SJM) ©SJM
ISSN 2534-9562
Volume 7 (2023) pp. 29-31
Received 7 May 2023. Published on-line 12 May 2023
web: http://www.sangaku-journal.eu/

The arbelos in Wasan geometry: the Aida arbelos and Problem 2023-1-5

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Abstract. Solving Problem 2023-1-5, we consider an Aida arbelos and its Archimedean circles, and give several Archimedean circles of the Aida arbelos.

Keywords. Aida arbelos, Archimedean circle

Mathematics Subject Classification (2010). 01A27, 51M04

1. Introduction

We consider the following problem proposed in [3] and cited in [4] (see Figure 1). A slight generalized problem of this can be found in [2].

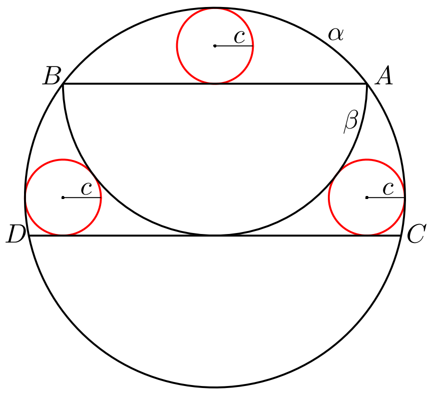

Figure 1.

Problem 1 ([3]). For a circle α of radius a, let AB and CD be parallel chords such that the semicircle β of diameter AB lying inside of α touches CD. If the two circles touching α internally β externally and the chord CD have radius c and are congruent to the maximal circle touching AB and the minor arc of α cut by AB, then show $a = 5c$.

The two circles touching the semicircle externally in the problem can be called Archimedean circles of a special arbelos called Aida arbelos ([6]), and the problem can immediately be solved if we consider an Aida arbelos. In this paper we introduce the Aida arbelos and give several Archimedean circles of the Aida arbelos.

2. AIDA ARBELOS AND PROBLEM 1

In this section we introduce the Aida arbelos (see Figure 2). For a point C on the line segment AB such that $c = |AB|/2$, $a = |BC|/2$ and $b = |CA|/2$, α, β and γ are the semicircles of diameters BC, CA and AB, respectively constructed on the same side of AB. The radical axis of the semicircles α and β is called the axis. The configuration consisting of the three semicircles α, β and γ is denoted by (α, β, γ) and is called an arbelos. Let R and S be the midpoints of the semicircles α and β, respectively. Let γ_{ad} be the semicircle concentric with γ passing through the points R and S such that $A_{ad}B_{ad}$ is the diameter of γ_{ad}, where the point A_{ad} lies on the same side of the axis as the point A. The configuration consisting of the three semicircles α, β and γ_{ad} was considered by Aida in [1], which is denoted by $(\alpha, \beta, \gamma_{ad})$ and is called an Aida arbelos. The Aida arbelos is a special case of a generalized arbelos called the arbelos with overhang ([5], [7]). The circle touching α (resp. β) externally γ internally and the axis from the side opposite to A (resp. B) has radius ab/c, and circles of the same radius are called Archimedean circles of (α, β, γ). The circle touching α (resp. β) externally γ_{ad} internally and the axis from the side opposite to A_{ad} (resp. B_{ad}) has radius ([1], [6])

$$r_{ad} = \frac{1}{2}(a + b - \sqrt{a^2 + b^2}), \tag{1}$$

and circles of the same radius are called Archimedean circles of $(\alpha, \beta, \gamma_{ad})$. Let O, P and Q be the centers of the semicircles γ, α and β, respectively.

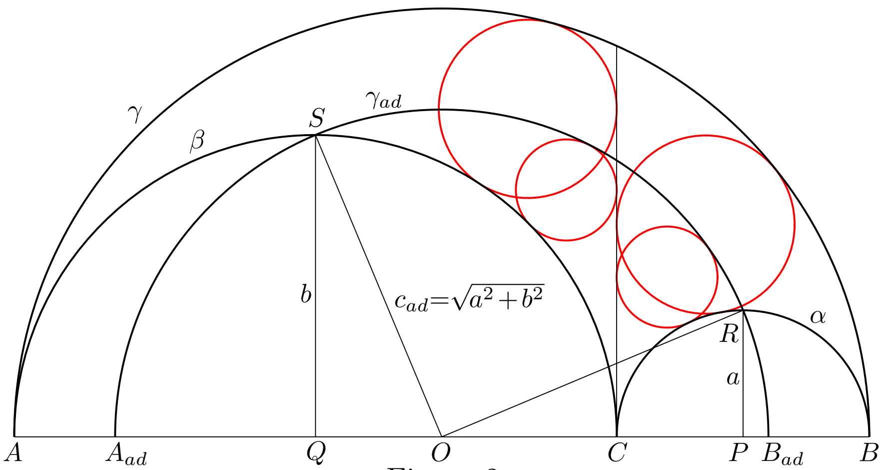

Figure 2.

We now solve Problem 1. Assume $2r_{ad} = |A_{ad}Q|$. Let c_{ad} be the radius of the semicircle γ_{ad}. Then $c_{ad} = \sqrt{a^2 + b^2}$. Since $c_{ad} = |A_{ad}Q| + a = |B_{ad}P| + b$ holds ([6]), $2r_{ad} = |A_{ad}Q|$ implies $a+b-\sqrt{a^2 + b^2} = \sqrt{a^2 + b^2} - a$. Hence we get $4a = 3b$, i.e., there is a real number z such that $a = 3z$ and $b = 4z$. Then we have $c_{ad} = 5z$ and $r_{ad} = z$.

3. Archimedean circles of $(\alpha, \beta, \gamma_{ad})$

We give several Archimedean circles of $(\alpha, \beta, \gamma_{ad})$ whose radius is given by (1) (see Figure 3).

Theorem 1. *The following statements are true.*
(i) *The incircles of the right triangles OPR and OQS are Archimedean circles of $(\alpha, \beta, \gamma_{ad})$. Let P' and Q' be the points such that $OPRP'$ and $OQSQ'$ are rectangles. Then the incircles of the triangles $OP'R$ and $OQ'S$ are Archimedean circles of $(\alpha, \beta, \gamma_{ad})$. Also we get four more triangles whose incircles are Archimedean circles of $(\alpha, \beta, \gamma_{ad})$ by rotating the two rectangles through $180°$ about the midpoint of RS.*
(ii) *Any circle touching γ internally and γ_{ad} externally is an Archimedean circle of $(\alpha, \beta, \gamma_{ad})$.*

Proof. The two right triangles are congruent with side lengths a, b and $\sqrt{a^2 + b^2}$. Therefore the inradius equals r_{ad}. This proves (i). The difference between the radii of the semicircles γ and γ_{ad} equals $2r_{ad}$. This proves (ii). □

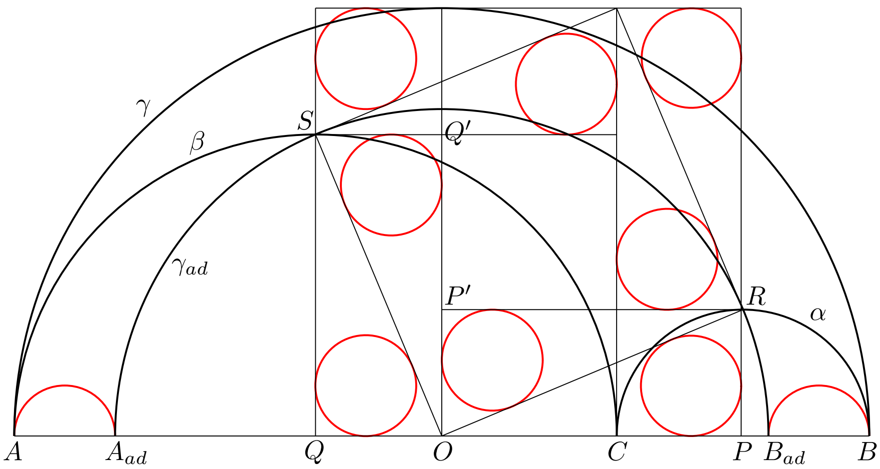

Figure 3.

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