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A configuration arising from Problem 2023-1-1

Hiroshi Okumura
Maebashi Gunma 371-0123, Japan
e-mail: hokmr@yandex.com

Abstract. We generalize Problem 2023-1-1 and consider a configuration of a rectangle and two pairs of congruent circles arising from the problem.

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1. Introduction

We consider the following problem proposed in [1] and cited in [2] (see Figure 1).

Problem 1 ([1])**.** For a rectangle $ABCD$, assume that α is the incircle of the triangle ABC, β is a circle in the triangle ACD and touching α and the side AC at their point of tangency and touching the side CD, γ is the circle touching β externally and the sides BC and CD from the inside of $ABCD$. If the circles β and γ are congruent, then show that the radius of α equals $|DA|/3$.

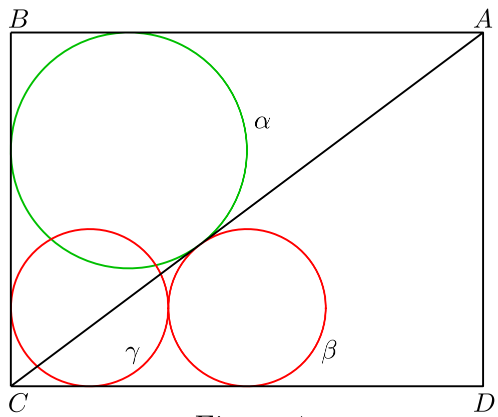

Figure 1.

In this paper we solve the problem in a general way, and consider a configuration of a rectangle and two pairs of congruent circles arising from the problem.

2. Generalization and a configuration $\mathcal{K}_n$

We consider the configuration consisting of the rectangle $ABCD$ and the circles α and β, which is denoted by $\mathcal{K}$. Let a and b be the radii of α and β, respectively, and let $p = |AB|$ and $q = |DA|$. Since α is the incircle of the right triangle ABC, we have

$$a = \frac{1}{2}\left(p + q - \sqrt{p^2 + q^2}\right). \tag{1}$$

Lemma 1. *The following relation holds for* $\mathcal{K}$*:*

$$b = \frac{2p^2 - pq + q^2 + (q - 2p)\sqrt{p^2 + q^2}}{2q}. \tag{2}$$

Proof. Let $\theta = \angle ACD/2$ (see Figure 2). Then $p = a + a\cot\theta$ and $q = a + b + (a + b)\cos 2\theta$ hold. Eliminating θ from the two equations and substituting (1) in the resulting equation, and solving the resulting equation for b, we get (2). □

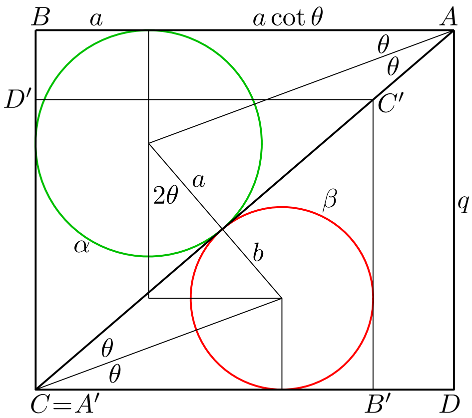

Figure 2: The configuration $\mathcal{K}$.

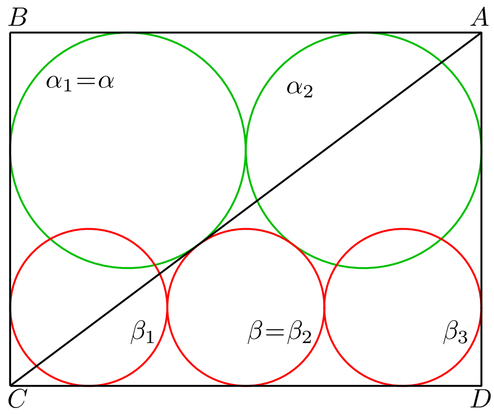

Figure 3: $\mathcal{K}_2$.

For an integer $n \geq 2$, assume that there are n congruent circles $\alpha_1, \alpha_2, \cdots, \alpha_n$ in the rectangle $ABCD$ and touching the side AB such that α_1 touches the side BC, α_2 touches α_1, and $\alpha_i (\neq \alpha_{i-2})$ touches α_{i-1} for $i = 3, 4, \cdots, n$, and α_n touches the side DA. Then the circles are called n *congruent circles on the side* AB. Moreover if $\alpha = \alpha_1, \alpha_2, \cdots, \alpha_n$ are n congruent circles on the side AB for the configuration $\mathcal{K}$, then $\mathcal{K}$ with the n congruent circles is denoted by $\mathcal{K}_n$ (see Figures 4 and 5, where the small red circles will be explained soon later). We will see that the figure in Problem 1 is a part of $\mathcal{K}_2$ (see Figure 3). The next theorem gives a generalized solution of Problem 1.

Theorem 1. *Let* $m = (n-1)(2n-1)$ *for an integer* $n \geq 2$. *The following statements are true.*
(i) $n < m$.
(ii) *For the configuration* $\mathcal{K}_n$ $(n \geq 2)$, *there are* m *congruent circles* $\beta_1, \beta_2, \cdots, \beta_n, \cdots, \beta_m$ *on the side* CD *such that* β_1 *touches the sides* BC *and* $\beta_n = \beta$.
(iii) *The following relations hold for* $\mathcal{K}_n$*:*

$$\frac{p}{q} = \frac{2n(n-1)}{2n-1}, \quad a = \frac{n-1}{2n-1}q, \quad b = \frac{n}{(2n-1)^2}q.$$

Proof. The part (i) follows from $m - n = (n-1)(2n-1) - n = 2(n-1)^2 - 1 \geq 2 - 1 > 0$. We prove (ii). Let $A'B'C'D'$ be the rectangle such that $A' = C$, B' and

C' lie on the sides CD and CA, respectively, and β is the incircle of the triangle $A'B'C'$ (see Figure 2). Then $ABCD$ and $A'B'C'D'$ are similar. Hence there are n congruent circles $\beta_1, \beta_2, \cdots, \beta_n = \beta$ on the side $A'B'$. Let m' be the positive real number such that $2m'b = p$, while we obviously have $2na = p$. We substitute (1) and (2) in the the last two equations and eliminate p, and solve the resulting equation for m'. Then we get $m' = (n-1)(2n-1)$. This proves (ii). Substituting (1) in $p = 2na$ and solving the resulting equation for p/q, we get the first equation of (iii). Substituting $p = 2na$ in the first equation, we get the second equation. The third equation follows from the second equation and $b = na/m$. □

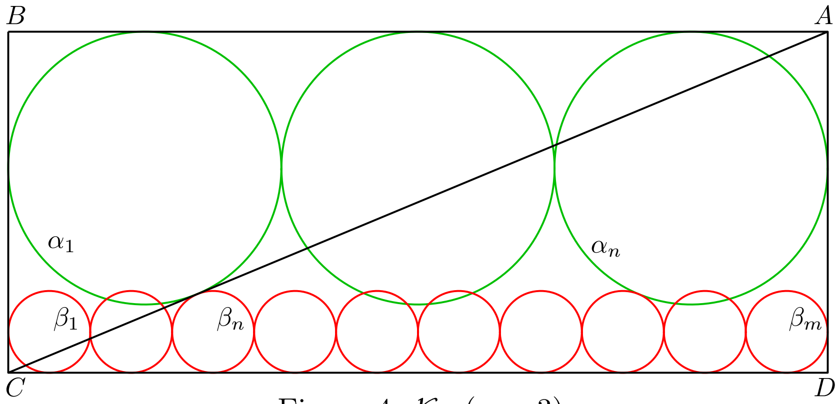

Figure 4: $\mathcal{K}_n$ $(n = 3)$.

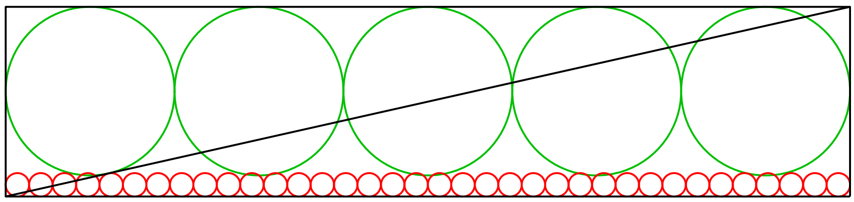

Figure 5: $\mathcal{K}_n$ $(n = 5)$.

The first equation of (iii) shows that the triangle ABC in $\mathcal{K}_n$ is a $(2n-1)$-$2n(n-1)$-$(2(n-1)n+1)$ triangle. Therefore ABC in Problem 1 is a 3-4-5 triangle.

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